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A COMPARATIVE STUDY OF DISCRETE TIME FILTERING FOR A NON-STATIONARY RANDOM INPUT

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A COMPARATIVE STUDY OF DISCRETE TIME FILTERING FOR A NON-STATIONARY RANDOM INPUT

by

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ABSTRACT

This thesis is concerned with a comparative study of discrete time filters using the theories of Wiener-Kolmogorov, Bode-Shannon, and Kalman, applied to the filtering of a non-stationary random signal in the presence of measurement noise. Programs are developed for the simulation of these systems and signals on a digital computer. Their filtering properties are compared for a random input signal with known steady state characteristics, starting at initial time $t = t_o$. The results show that when the gain of the Kalman filter is equal to its steady state value, the Kalman and Wiener-Kolmogorov filters perform identically. For large initial errors the Kalman filter, with large initial gain, gives the best transient response; for small initial errors the Kalman and Wiener-Kolmogorov filters are essentially equivalent in their transient responses.

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ACKNOWLEDGEMENT

I wish to take this opportunity to express my appreciation for the assistance and encouragement given me by Dr. Sydney R. Parker in the writing of this thesis.

CHAPTER I

INTRODUCTION

1. OBJECTIVE

This thesis examines and compares several techniques for the filtering of random signals in the presence of measurement noise. In diagram form, the problem is as shown in Figure 1; $x(t)$ is the random signal to be estimated, $v(t)$ is a random measurement noise, $\hat{x}(t)$ is the estimated or filtered signal, and $e(t)$ is the error in estimation. The criteria for the optimum filter is one which reduces the mean square error to a minimum. The following techniques are studied and compared with a common example:

(1) Discrete Wiener-Kolmogorov filter

(2) Discrete Kalman filter

(3) Bode-Shannon discrete time filter

For all of the filters it is assumed that the statistical properties of the signal and the noise are known by their correlation functions, or their power density spectra, or by equivalent white noise excited dynamic models. It is also assumed that the signal and the measurement noise are uncorrelated. Wiener-Kolmogorov theory assumes that the signal and the noise are stationary; that is, their statistical properties do not vary with time. This implies that all signals are initiated at $t = -\infty$. Kalman and Bode-Shannon filtering, on the other hand, assume that signals start at some initial time, t_0, and that as time progresses the stationary probability characteristics of the signals are established. In addition, Kalman (as

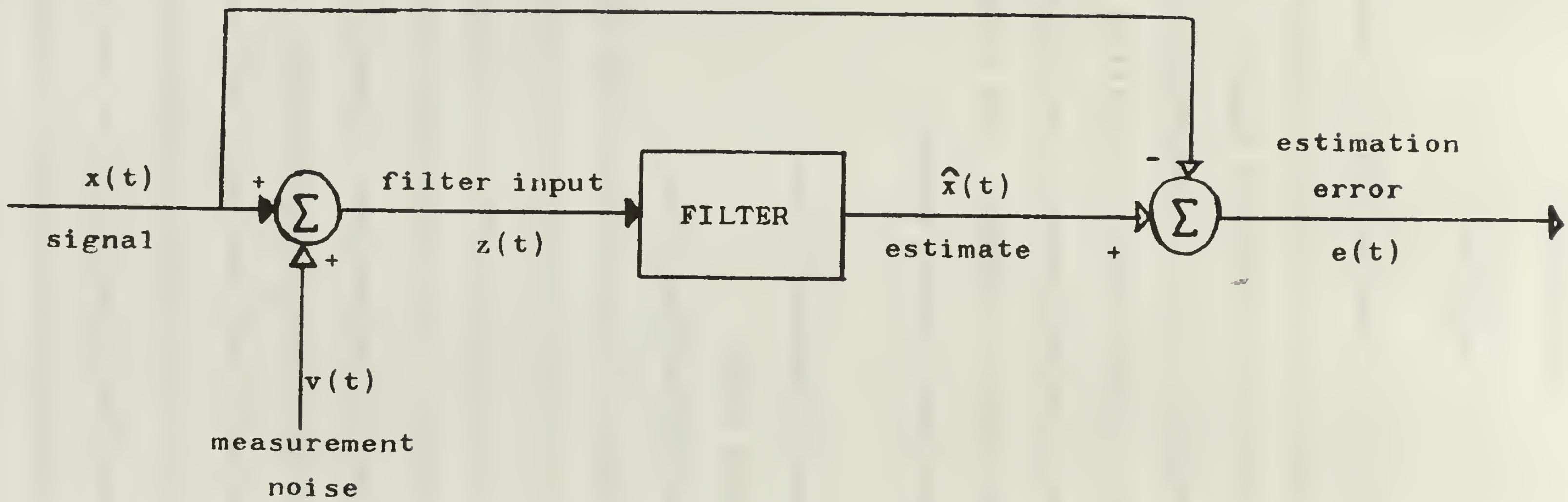

FIGURE 1

BASIC FILTER PROBLEM

well as Bode-Shannon) assumes that the random signal process is Markovian, that is, can be thought of as generated by a linear dynamic system excited by white noise.* During the transient stage, from $t = t_0$ until the statistical properties are established, the Kalman and Bode-Shannon filters behave as time-varying devices. In the steady state, all of the filters can be expected to behave identically since they each establish a minimum mean square error estimate.

In order to test and compare the operation of these filters, particularly for a non-stationary input, the Wiener-Kolmogorov and Kalman filters have been designed for discrete time simulation on a digital computer for filtering the same random input signal and measurement noise. The output of each filter and the actual random input signal have been compared starting at time $t = t_0$ until steady state conditions are established.

The type of filtering problem studied can be compared with a basic tracking problem where a target, with statiscally known maneuvering properties, suddenly appears. It is desired to track the target and, in particular, to investigate how long it takes the output to reach the steady state minimum mean squared error. The results of this investigation are presented in this thesis together with a summary of pertinent background theory. In addition, the discrete time formulations for the Wiener-Kolmogorov theory have been developed and programmed for the specific example treated.

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*R. E. Kalman, New Methods and Results in Linear Prediction and Estimation Theory, Technical Report No. 61-1, (Baltimore, Maryland: Research Institute for Advanced Study, 1961) p. 3.

The basic theory for discrete Kalman and Bode-Shannon filtering appears in the literature and has been adapted to the example. Programs necessary for generating the stochastic signals from white noise have also been developed.

2. HISTORICAL BACKGROUND

The many present day theories for the filtering of time sequences are basically derived from the original work of Wiener and Kolmogorov published in 1941 and 1942.[2,14,17] They obtained the direct solution of the filter problem independently and simultaneously by the use of the calculus of variations. This approach led to the filter's impulse response being characterized by the solution of a difficult integral equation known as the Wiener-Hopf equation.

One of the next major developments in the solution of the filtering problem was accomplished by Bode and Shannon[2,17] in 1950 with the simplification of the derivation of the Wiener-Kolmogorov filter. They used frequency domain analysis and conventional circuit theory concepts to interpret mathematical operations in physically intuitive terms. Bode and Shannon also applied this frequency domain technique to discrete signals with constant sampling intervals.

In most cases, the solution of the Wiener-Hopf integral equation is a formidable task. With the advent of the digital computer, an alternate approach was sought to avoid this difficulty. In 1960 Kalman[9,10,11] and Bucy[11] attacked the filtering problem from the state variable point of view. They evolved a set of differential

equations, equivalent to the Wiener-Hopf integral equation, whose solution yields the optimum filter. These differential equations may be synthesized in a sequential fashion and therefore digital computer solutions easily obtained.

CHAPTER II

THE WIENER-KOLMOGOROV FILTER

There are several methods for deriving the continuous Wiener-Kolmogorov filter. Appendix A summarizes the solution which leads to the Wiener-Hopf integral equation.#

$$\varphi_{zx}(t) = \int_{-\infty}^{\infty} H(\tau)\, \varphi_{zz}(t - \tau)d\tau, \quad t \geq 0 \qquad (2.1)$$

$H(\tau)$ is the desired filter impulse response, $\varphi_{zz}(t)$ is the auto-correlation function of the signal plus noise, and $\varphi_{zx}(t)$ is the crosscorrelation function of the signal plus noise, and the signal.

The requirements that must be fulfilled to use Eq. 2.1 are (1) that all signals are stationary and exist for all time; (2) that the signal and the noise are uncorrelated. In order to solve the equation, the autocorrelation function of the signal and the auto-correlation function of the noise must be known. These may be obtained by taking the Fourier transform of the respective power spectral densities, as given by the Wiener-Khintchine equations:*

$$W_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_{xx}(\tau)e^{-j\omega\tau}d\tau = \frac{1}{2\pi} \mathcal{F} \left[\varphi_{xx}(\tau)\right]$$

#George C. Newton, Jr., Leonard A. Gould, and James F. Kaiser, Analytical Design of Linear Feedback Controls (New York: John Wiley and Sons, Inc., 1957) p. 144.

*John L. Stewart, Fundamentals of Signal Theory (New York: McGraw-Hill Book Company, Inc., 1960) p. 306.

$$\varphi_{xx}(\tau) = \int_{-\infty}^{\infty} W_x(\omega) e^{+j\omega\tau} \, d\omega = 2\pi \mathcal{F}^{-1} \left[W_x(\omega) \right]$$

where $W_x(\omega)$ is the power density spectrum of $x(t)$. Even when sufficient knowledge of the signal is available, the solution of Eq. 2.1 is generally an extremely difficult task.

1. SOLUTION BY SPECTRUM FACTORIZATION

The solution of Eq. 2.1 using the technique known as spectrum factorization has been developed by several authors.[2,14,17] The autocorrelation function of the signal plus noise, $\varphi_{zz}(s)$, is factored so that

$$\varphi_{zz}(s) = \varphi^{+}_{zz}(s) \, \varphi^{-}_{zz}(s) \tag{2.1-1}$$

where $\varphi^{+}_{zz}(s)$ contains those factors of $\varphi_{zz}(s)$ which have poles and zeros in the left half plane. The term

$$\left[\frac{\varphi_{zx}(s)}{\varphi^{-}_{zz}(s)} \right]_{+} \tag{2.1-2a}$$

is defined by those terms of the partial fraction expansion of:

$$\frac{\varphi_{zx}(s)}{\varphi^{-}_{zz}(s)}$$

which have poles in the left half of the s-plane. That is

$$\frac{\varphi_{zx}}{\varphi^{-}_{zz}} = \left[\frac{\varphi_{zx}(s)}{\varphi^{-}_{zz}(s)} \right]_{+} + \left[\frac{\varphi_{zx}(s)}{\varphi^{-}_{zz}(s)} \right]_{-} \tag{2.1-2b}$$

The resulting weighting or transfer function of the filter is given by:

$$H(s) = \frac{\left[\dfrac{\varphi_{zx}(s)}{\varphi^{-}_{zz}(s)}\right]_{+}}{\varphi^{+}_{zx}(s)} \qquad (2.1\text{-}3)$$

The use of this equation may be demonstrated by the example shown in Figure 1. Consider that x(t) has an autocorrelation function:

$$\varphi_{xx}(\tau) = \frac{1}{2} e^{-|\tau|} \qquad (2.1\text{-}4)$$

The Laplace transform of the autocorrelation function is then

$$\varphi_{xx}(s) = \frac{1}{1 - s^2} \qquad (2.1\text{-}5)$$

which yields the power density spectrum,

$$W_x(\omega) = \frac{1}{2\pi} \left[\frac{1}{1 + \omega^2}\right] \qquad (2.1\text{-}6)$$

This signal can be shown to correspond to the random walk signal shown in Figure 2, where x(t) is a rectangular wave with values $+\frac{1}{4}$ or $-\frac{1}{4}$ and with zero crossings located at event points which are Poisson distributed with an average frequency of $\frac{1}{2}$ crossings per second. This time domain signal is not unique since many signals may have the same power density spectrum.

Now consider a white noise source where,

$$\varphi_{vv}(\tau) = \delta(\tau) \qquad (2.1\text{-}7)$$

where $\delta(\tau)$ is the Dirac delta function, which has the following Laplace transform:

$$\varphi_{vv}(s) = 1 \qquad (2.1\text{-}8)$$

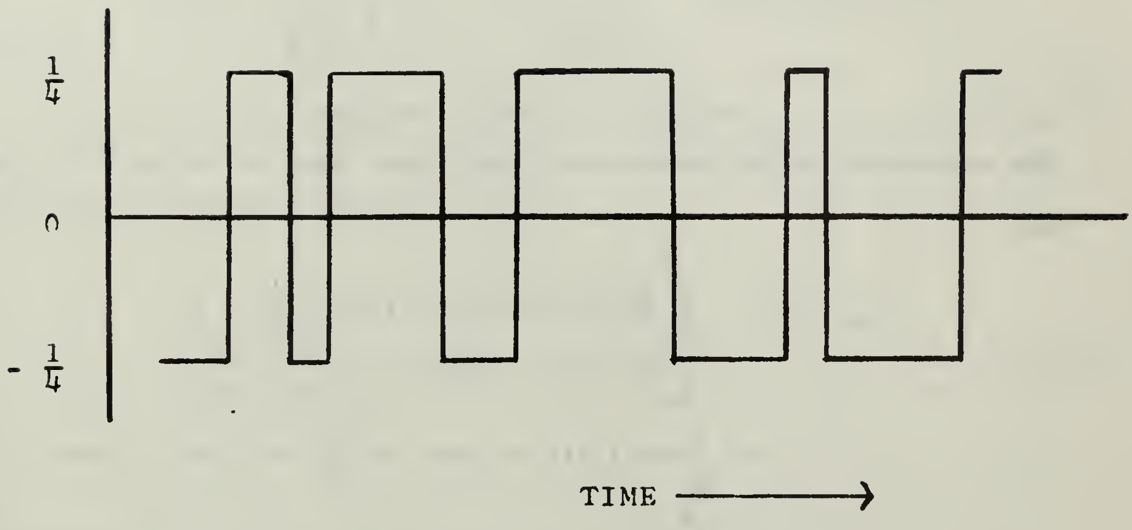

FIGURE 2

RANDOM WALK SIGNAL

The autocorrelation function of the signal plus noise is

$$\varphi_{zz}(s) = \varphi_{xx}(s) + \varphi_{vv}(s) + \varphi_{vx}(s) + \varphi_{xv}(s) \qquad (2.1\text{-}9)$$

Since the signal and noise are assumed to be uncorrelated the last two terms are zero and $\varphi_{zz}(s)$ reduces to:

$$\varphi_{zz}(s) = \varphi_{xx}(s) + \varphi_{vv}(s)$$

$$= \frac{1}{1 - s^2} + 1 = \frac{2 - s^2}{1 - s^2} \qquad (2.1\text{-}10)$$

The crosscorrelation function of the signal plus noise, and the signal is:

$$\varphi_{zx}(\tau) = \int_{-\infty}^{\infty} (v(t) + x(t)) \; x(t + \tau)dt$$

$$= \int_{-\infty}^{\infty} v(t) \; x(t + \tau)dt + \int_{-\infty}^{\infty} x(t) \; x(t + \tau)dt$$

$$= \int_{-\infty}^{\infty} x(t) \; x(t + \tau)dt = \varphi_{xx}(\tau) \qquad (2.1\text{-}11)$$

Since the noise and the signal are uncorrelated,

$$\varphi_{zx}(s) = \frac{1}{1 - s^2} \qquad (2.1\text{-}12)$$

Factoring Eq. 2.1-10 results in:

$$\varphi_{zz}(s) = \frac{2 - s^2}{1 - s^2} = \frac{(\sqrt{2} + s) \quad (\sqrt{2} - s)}{(1 + s) \quad (1 - s)} \qquad (2.1\text{-}13)$$

Therefore, from Eq. 2.1-1,

$$\varphi^{+}_{zz}(s) = \frac{\sqrt{2} + s}{1 + s}$$

and,

$$\varphi^{-}_{zz}(s) = \frac{\sqrt{2} - s}{1 - s}$$

Now,

$$\frac{\varphi_{zx}(s)}{\varphi^{-}_{zz}(s)} = \frac{1}{1 - s^2} \quad \frac{1 - s}{(\sqrt{2} - s)} = \frac{1}{(1 + s) \quad (\sqrt{2} - s)}$$

$$= \frac{1}{2.414} \left(\frac{1}{\sqrt{2} - s} + \frac{1}{1 + s} \right) \qquad (2.1\text{-}14)$$

Therefore, from Eq. 2.1-2a,

$$\left[\frac{\varphi_{zx}(s)}{\varphi^{-}_{zz}(s)} \right]_{+} = \frac{1}{2.414} \left(\frac{1}{1 + s} \right) \qquad (2.1\text{-}15)$$

Substituting into Eq. 2.1-3,

$$H(s) = \frac{1}{2.414} \left(\frac{1}{1 + s} \right) \left(\frac{1 + s}{\sqrt{2} + s} \right)$$

$$= \frac{1}{2.414} \left(\frac{1}{\sqrt{2} + s} \right) \qquad (2.1\text{-}16)$$

The resulting filter weighting function, or impulse response, is:

$$H(t) = \begin{cases} \frac{1}{2.414} \, e^{-\sqrt{2}\, t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \qquad (2.1\text{-}17)$$

This filter is physically realizable, since it is causal.

2. THE BODE-SHANNON SOLUTION

The frequency domain technique for the Wiener-Kolmogorov filter as developed by Bode and Shannon,[#] is to convert the signal power spectral density to that of white noise, and then have a filter operate on this white noise. The development of this technique appears in several references [2,14,17] and is discussed in Appendix B. The following results are obtained.

Given $W_x(\omega)$, the power spectral density of the signal x(t); and $W_v(\omega)$, the power spectral density of the noise v(t), consider the transfer function of the following minimum phase (zeros and poles in the left half of the s-plane) filter:

$$B(\omega) = \sqrt{W_x(\omega) + W_v(\omega)} \tag{2.2-1}$$

From a white noise input this filter will produce a power spectral density equal to that of the signal plus noise. As shown in Appendix B, the following transfer function can then be obtained for the non-causal optimal filter.

$$h_\omega(\omega) = \frac{W_x(\omega)\ B(\omega)}{[W_x(\omega) + W_v(\omega)]} \tag{2.2-2}$$

The preceding equations are represented in Figure 3. Since $h_\omega(\omega)$ is non-causal, let

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[#]Ralph J. Schwarz and Bernard Friedland, <u>Linear Systems</u> (New York: McGraw-Hill Book Company, 1965), p. 334-45.

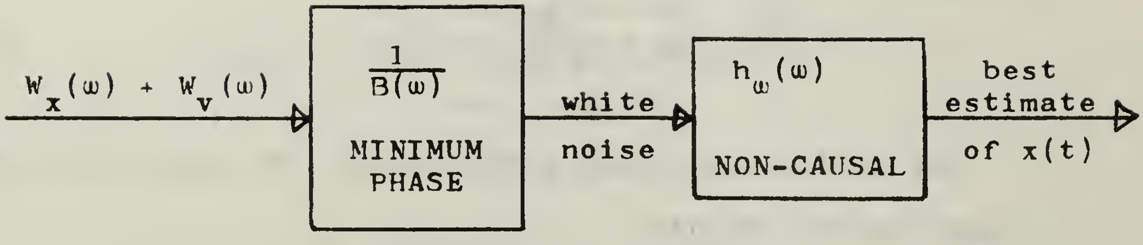

FIGURE 3

BODE-SHANNON THEORY

$$g_{\omega}(t) = \begin{cases} 0 & , t < 0 \\ h_{\omega}(t) & , t \geq 0 \end{cases} \tag{2.2-3}$$

where $g_{\omega}(t)$ represents the portion of $h_{\omega}(t)$ which exists for $t \geq 0$. This is equivalent to the spectral factorization of Eq. 2.1-1 and Eq. 2.1-2a. The desired optimum causal filter has the transfer function

$$H(\omega) = \frac{G_{\omega}(\omega)}{B\ (\omega)} \tag{2.2-4}$$

Now consider the example problem again. The signal, $x(t)$, has a power spectral density

$$W_x(\omega) = \frac{1}{2\pi} \left[\frac{1}{1 + \omega^2}\right] \tag{2.2-5}$$

and the uncorrelated measurement noise is white, with a power spectral density

$$W_v(\omega) = \frac{1}{2\pi} \tag{2.2-6}$$

The power spectral density of the signal plus noise is therefore

$$W_z(\omega) = W_x(\omega) + W_v(\omega) = \frac{1}{2\pi} \left[\frac{1}{1 + \omega^2} + 1\right] = \frac{1}{2\pi} \left[\frac{2 + \omega^2}{1 + \omega^2}\right] \tag{2.2-7}$$

This may be factored

$$W_z(\omega) = \frac{1}{2\pi} \frac{(\sqrt{2} + j\omega)\ (\sqrt{2} - j\omega)}{(1 + j\omega)\ \ (1 - j\omega)} \tag{2.2-8}$$

Thus, the minimum phase transfer function required to produce this spectrum from white noise, using Eq. 2.2-1, is:

$$B(\omega) = \frac{1}{\sqrt{2\pi}} \quad \frac{\sqrt{2} + j\omega}{1 + j\omega} \tag{2.2-9}$$

The inverse of this transfer function will produce white noise from the signal plus noise.

Now, substituting in Eq. 2.2-2,

$$h_\omega(\omega) = \frac{\frac{1}{2\pi}\left(\frac{1}{1+\omega^2}\right)\left(\frac{\sqrt{2}+j\omega}{1+j\omega}\right) \frac{1}{\sqrt{2\pi}}}{\frac{1}{2\pi} \quad \frac{2+\omega^2}{1+\omega^2}}$$

$$= \frac{\sqrt{2} + j\omega}{(2+\omega^2)\ (1+j\omega)} \quad \frac{1}{\sqrt{2\pi}}$$

$$= \frac{1}{(\sqrt{2} - j\omega)\ (1+j\omega)} \ \frac{1}{\sqrt{2\pi}}$$

$$= \frac{1}{2.414\ \sqrt{2\pi}} \left(\frac{1}{\sqrt{2} - j\omega} + \frac{1}{1+j\omega} \right) \tag{2.2-10}$$

By taking the inverse Fourier transform of Eq. 2.2-10, the impulse response, or weighting function, is then

$$h_\omega(t) = \begin{cases} \frac{1}{\sqrt{2\pi}} \quad \frac{1}{2.414} \quad e^{\sqrt{2}\ t} & , t < 0 \\ \frac{1}{\sqrt{2\pi}} \quad \frac{1}{2.414} \quad e^{-t} & , t \geq 0 \end{cases} \tag{2.2-11}$$

which is non-causal. The best causal impulse response, from Eq. 2.2-3, is

$$g_\omega(t) = \begin{cases} 0 & , \ t < 0 \\ \frac{1}{\sqrt{2\pi}} \frac{1}{2.414} e^{-t} & , \ t \geq 0 \end{cases} \tag{2.2-12}$$

The desired impulse response is therefore,

$$H(\omega) = \frac{G_\omega(\omega)}{B(\omega)}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{2.414}\right) \left(\frac{1}{1 + j\omega}\right) \left(\frac{1 + j\omega}{\sqrt{2} + j\omega}\right) \sqrt{2\pi}$$

$$= \frac{1}{2.414} \frac{1}{\sqrt{2} + j\omega}$$

and

$$H(t) = \begin{cases} 0 & , \ t < 0 \\ \frac{1}{2.414} e^{-\sqrt{2}\, t} & , \ t \geq 0 \end{cases} \tag{2.2-13}$$

This filter impulse response is identical to Eq. 2.1-17, as expected.

3. DISCRETE TIME SOLUTION OF THE WIENER-KOLMOGOROV FILTER

The initial problem to be solved for digital computer simulation of the Wiener-Kolmogorov filter is the realization of the discrete signal, x(kT), based on the required power spectral density

$$W_x(\omega) = \frac{1}{2\pi} \left[\frac{1}{1 + \omega^2}\right] \tag{2.3-1}$$

Since the power density spectrum of white noise is a constant, this power spectral density may be obtained by driving a linear system, $G_1(j\omega)$, by white noise, w(t), so that

$$W_x(\omega) = G_1(j\omega)\, G_1(-j\omega)\, W_w(\omega) = \frac{1}{2\pi} \left[\frac{1}{1 + \omega^2}\right] \qquad (2.3\text{-}2)$$

where

$$G_1(j\omega) = \frac{1}{1 + j\omega}$$

It is extremely difficult, if not impossible, to obtain a discrete white noise source which is flat over all frequency. In practice, what needs to be done is to make the spectrum of the noise source flat over the bandwidth of $G_1(j\omega)$. The 3 decibel cutoff frequency for $G_1(j\omega)$ is one radian per second. The effective noise bandpass is given by

$$B = \frac{\pi}{2}\,(1) = \frac{\pi}{2} \text{ radian/sec.} = 1/4 \text{ Hz.} \qquad (2.3\text{-}3)$$

where noise bandwidth is $\frac{\pi}{2}$ times the 3 decibel cutoff frequency.*
The sampling rate for the discrete noise should be much greater than 1/4 Hz., and 100 Hz. was chosen as convenient. The sampling period, T, therefore is 0.01 seconds. The spectrum of the noise should then be flat to 50 Hz. as shown in Figure 4.

The autocorrelation function of this band limited white noise is then given by

$$\varphi_{ww}(\tau) = \frac{100 \sin (100\, \pi\tau)}{100\, \pi\tau} \qquad (2.3\text{-}4)$$

as shown in Figure 5. The variance of w(t), the noise sequence, is given by

$$\varphi_{ww}(0) = \sigma_w^2 = 100 = \frac{1}{T} \qquad (2.3\text{-}5)$$

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*Mischa Schwartz, Information Transmission, Modulation, and Noise (New York: McGraw-Hill Book Company, Inc., 1959), pp. 206-207.

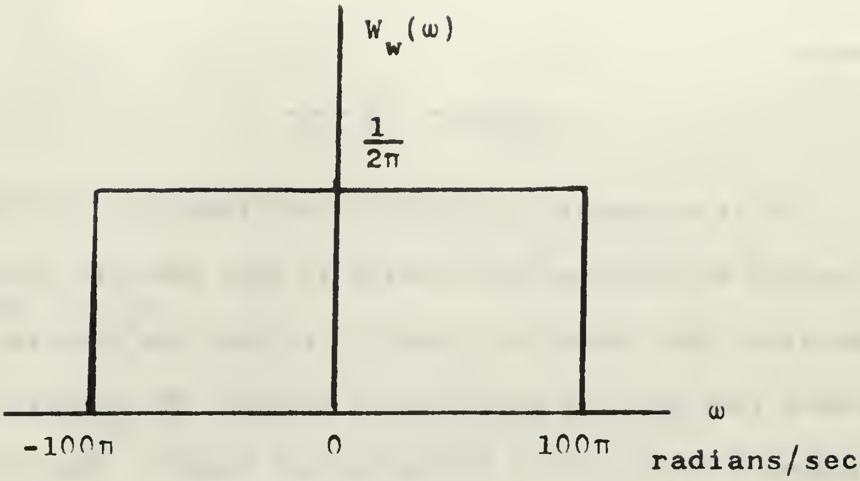


FIGURE 4

WHITE NOISE SPECTRUM

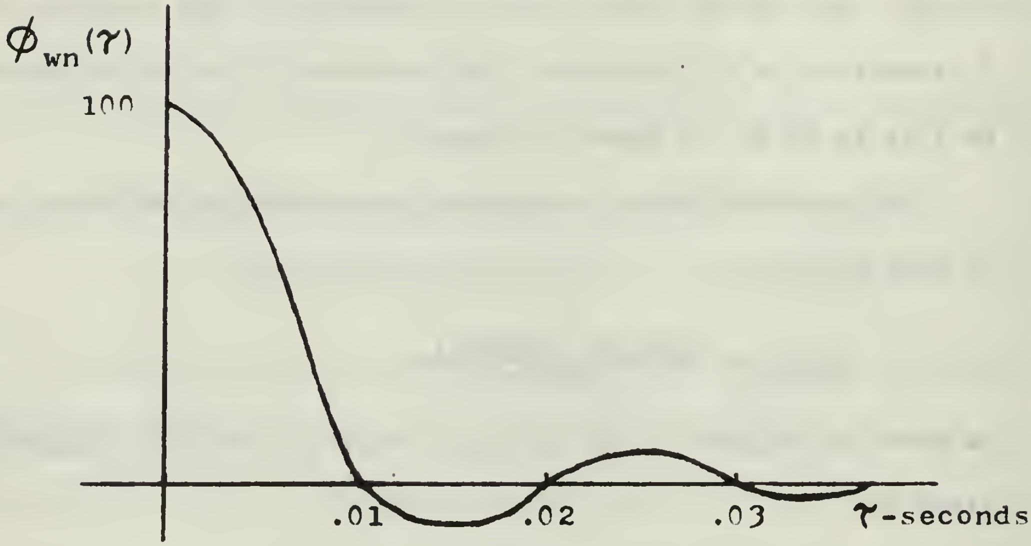


FIGURE 5

AUTOCORRELATION OF WHITE NOISE SOURCE

The white noise source is then simulated by a series of pulses of repetition period .01 seconds, whose amplitudes have a Gaussian distribution with zero mean and variance 100, followed by a zero order hold as shown in Figure 6. The recursive equation necessary for the computer simulation of the desired signal source is derived by taking the Z transform of the input-output transfer function of Figure 6.

$$\frac{x(z)}{w(z)} = Z\left[\left(\frac{1}{s+1}\right)\left(\frac{1-e^{-sT}}{s}\right)\right]$$

$$= \frac{1-e^{-T}}{z-e^{-T}} \qquad (2.3\text{-}6)$$

Hence

$$x(z)\,(z - e^{-T}) = w(z)\,(1 - e^{-T})$$

and

$$zx(z) = x(z)e^{-T} + w(z)\,(1 - e^{-T})$$

Therefore,

$$x(k+1) = e^{-T}x(k) + (1 - e^{-T})\,w(k);\; T = .01 \text{ sec.} \qquad (2.3\text{-}7)$$

To check that the signal does, in fact, have the desired power spectral density, a subroutine entitled HARM[#] was used to obtain the Fast Fourier Transform. The Fourier coefficients were then squared and plotted versus radian frequency, and the graph, Figure 7, obtained. Included on this graph is the desired power spectral density curve.

[#] International Business Machines Corporation, <u>System /360 Scientific Subroutine Package, (360A-CM-03X) Version II Programmer's Manual</u>, (White Plains, New York: International Business Machines Corporation, 1967)

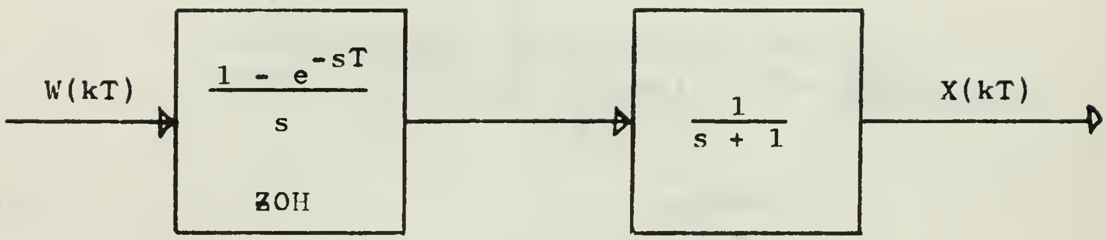
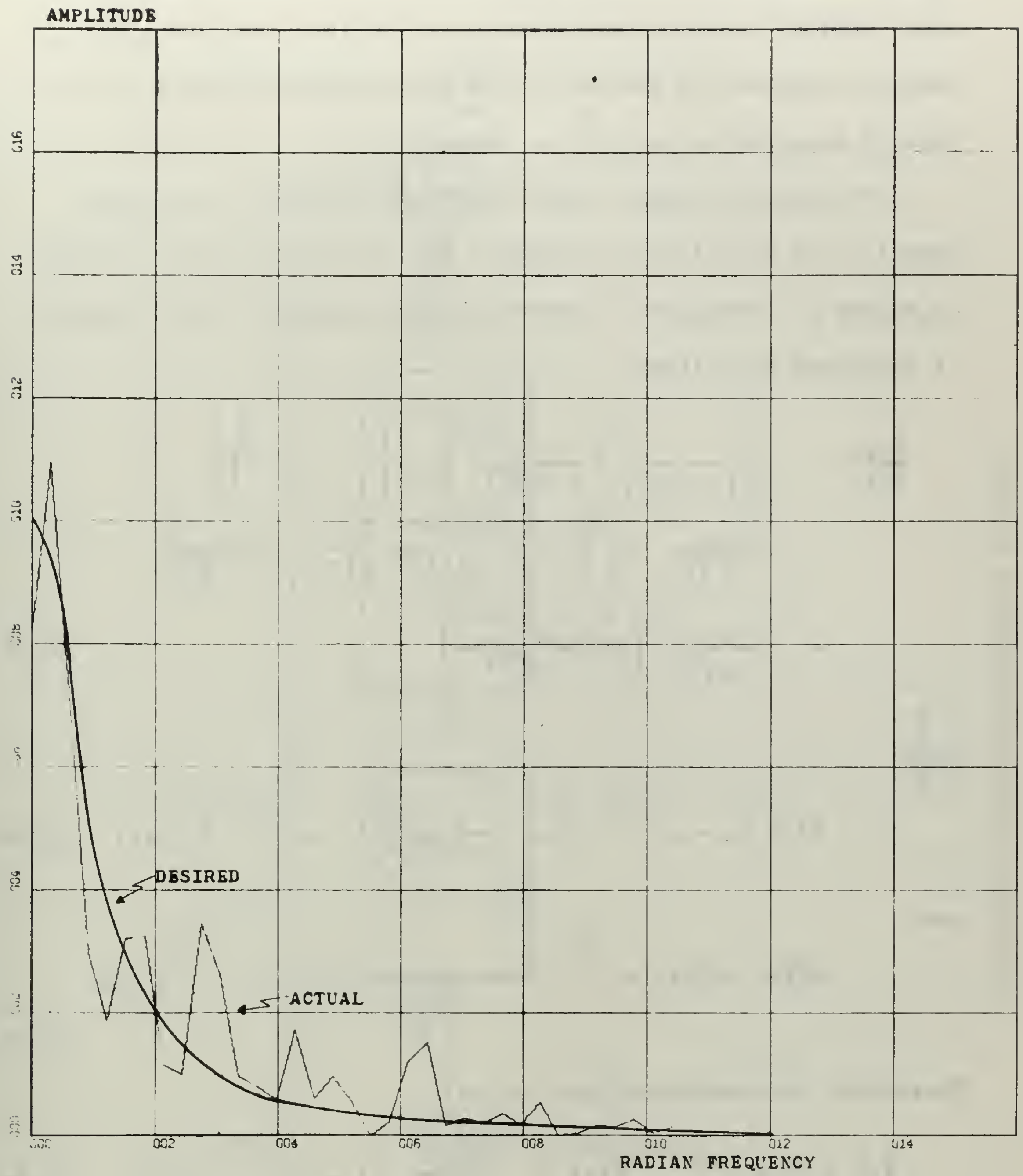


FIGURE 6

SIGNAL SOURCE



X-SCALE=2.00E+00 UNITS INCH.
Y-SCALE=2.00E-01 UNITS INCH.
P.S.D. OF SIGNAL GENERATOR
T=0.01 FLETCHER H.G. THESIS

FIGURE 7

The program was run for 2048 samples and it can be seen that the power density spectrum approximates the desired spectrum. The computer program for generating the signal and checking its power density spectrum is included as Appendix C.

Now that the proper signal has been obtained, a recursive equation for the filter is needed. The filtering system is drawn in Figure 8. Using the Z transform, the recursive filter equation is developed as follows:

$$\begin{aligned}
\frac{\hat{x}(z)}{z(z)} &= Z\left[\left(\frac{1}{1+\sqrt{2}}\right)\left(\frac{1}{s+\sqrt{2}}\right)\left(\frac{1}{s}\right)\left(1-e^{-sT}\right)\right] \\
&= \frac{1}{1+\sqrt{2}}\; Z\left[\left(1-e^{-sT}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{s}-\frac{1}{s+\sqrt{2}}\right)\right] \\
&= \frac{1}{2+\sqrt{2}}\left(\frac{1-e^{-\sqrt{2}\,T}}{z-e^{-\sqrt{2}\,T}}\right) \qquad (2.3\text{-}8)
\end{aligned}$$

Hence

$$\hat{x}(z)\left(z-e^{-\sqrt{2}\,T}\right) = \frac{1}{2+\sqrt{2}}\left(1-e^{-\sqrt{2}\,T}\right) z(z) \qquad (2.3\text{-}9)$$

and

$$z\hat{x}(z) = \hat{x}(z)\, e^{-\sqrt{2}\,T} + \frac{1}{2+\sqrt{2}}\left(1-e^{-\sqrt{2}\,T}\right) Z(k) \qquad (2.3\text{-}10)$$

Therefore, the recursive equation is:

$$\hat{X}(k+1) = e^{-\sqrt{2}\,T}\,\hat{X}(k) + \frac{1}{2+\sqrt{2}}\left(1-e^{-\sqrt{2}\,T}\right) Z(k) \qquad (2.3\text{-}11)$$

The error is

$$E(k) = \hat{X}(k) - X(k) \qquad (2.3\text{-}12)$$

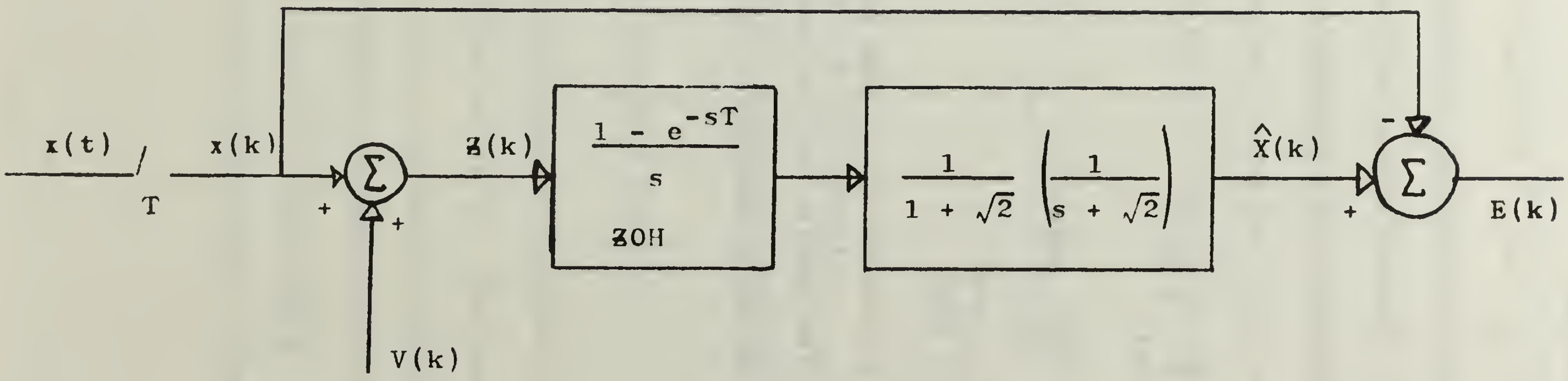


FIGURE 8

DISCRETE WIENER-KOLMOGOROV FILTER

The assumption is made that $\hat{X}(0)$ is zero. The sampling interval is 0.01 seconds. The measurement noise, v(t), is white, with a power spectral density, $W_v(\omega) = \frac{1}{2\pi}$, and is developed in an identical fashion to w(t), the driving noise for the signal generator. The two noise sources are uncorrelated. The mean square error and the variance of the error are then calculated, on a continuing basis, using the equations:

$$\overline{E(n)} = \frac{1}{n} \sum_{k=1}^{n} E(k) \tag{2.3-13}$$

$$\overline{E(n)^2} = \frac{1}{n} \sum_{k=1}^{n} (E(k))^2 \tag{2.3-14}$$

$$\sigma_e = \overline{E^2(k)} - \overline{E(k)}^2 \tag{2.3-15}$$

The computer run was for 30 seconds, i.e., 3000 samples, so that steady state is reached. The results are discussed in Chapter V. The computer program for the Wiener-Kolmogorov filter appears as Appendix D.

CHAPTER III

BODE-SHANNON DISCRETE TIME FILTERING

The Bode and Shannon frequency domain approach may be applied in the time domain directly, as developed by Schwarz and Friedland.[17] The formulation is now in terms of summations, discrete signals, and discrete power spectral densities. This technique results in the following equation which is similar to the Wiener-Hopf integral equation.

$$\varphi_{zx}(k,n) = \sum_{j=0}^{n} h(n,j)\, \varphi_{zz}(j,k) \qquad (3.1a)$$

$$k = 0, 1, 2, \ldots n$$

where

$$\varphi_{zx}(k,n) = E\left[z(k)\ x(n)\right] \qquad (3.1b)$$

is the discrete crosscorrelation function between $z(k)$ and $x(n)$, and

$$\varphi_{zz}(j,k) = E\left[z(j)\ z(k)\right] \qquad (3.1c)$$

is the discrete autocorrelation function of $z(k)$. The unit impulse response of the filter is $h(n,j)$. The notation $h(n,j)$ denotes the response at time n due to an impulse applied at time j where $j \le n$. Since the impulse response of the filter must be causal, this requires that $h(n,j)$ equal zero for n less than j. Equation 3.1 holds for both stationary and non-stationary signals. When the signals are stationary then

$$\varphi_{zx}(k,n) = \varphi_{zx}(n - k) \qquad (3.2a)$$

$$\varphi_{zz}(j,k) = \varphi_{zz}(k - j) \qquad (3.2b)$$

$$h(n,j) = h(n - j) \tag{3.2c}$$

and Eq. 3.1 reduces to

$$\varphi_{zx}(i) = \sum_{m=0}^{\infty} h(m)\, \varphi_{zz}(m - i) \tag{3.2d}$$

where

$$i = n - k \quad \text{and } m = n - j \tag{3.2e}$$

The power spectral density of a discrete signal, $z(kT)$, is defined as:*

$$\begin{aligned} W_z'(z) &= T\, Z \left[\varphi_{zz}(nT)\right] \\ &= T \sum_{n=-\infty}^{\infty} \varphi_{zz}(nT)\, z^{-n} \end{aligned} \tag{3.3}$$

This is the discrete form of one of the Wiener-Khintchine equations. The prime denotes the power spectral density of a discrete signal. It should be noted that the z variable within the parenthesis denotes the Z-transform shifting variable. Proceeding as in Appendix B, a transfer function, $B(z)$, is developed which will convert a signal with a power spectral density of $W_z'(z)$ to a white noise process, $u(kT)$. This requires that

$$B(z)\, B(z^{-1})\, W_z'(z) = 1 \tag{3.4}$$

Since $B(z)$ is causal, its Z transform must be given by

$$B(z) = \sum_{n=0}^{\infty} b(n)\, z^{-n} \tag{3.5}$$

---

*Schwarz and Friedland[17] use this definition. Kuo[12] uses a factor of $1/T$ instead of T.

If $W_z^{'}(z)$ can be factored into a product,

$$W_z^{'}(z) = \overline{W}_z(z)\ \overline{W}_z(z^{-1}) \tag{3.6}$$

then

$$B(z) = \frac{1}{\overline{W}_z(z)} \tag{3.7}$$

For a white noise input $\varphi_{uu}(j,k)$ equals $\delta(j,k)$, and Eq. 3.1 reduces to

$$\begin{cases} \varphi_{ux}(k,n) = h(n,j) & \text{for } j \leq n \\ \varphi_{ux}(k,n) = 0 & \text{for } j > n \end{cases} \tag{3.8}$$

The impulse response of an optimal causal filter for a white noise input, $u(kT)$, and a desired output, $X(kT)$, is then given by

$$g(n,j) = \varphi_{ux}(k,n) \tag{3.9}$$

For stationary processes Eq. 3.9 reduces to

$$g(m) = \varphi_{ux}(m) \qquad \text{for } m = n - k > 0 \tag{3.10}$$

Taking the Z transform of Eq. 3.10, with the aid of Eq. 3.3, yields

$$g(z) = \left[\frac{W_{ux}^{'}(z)}{T}\right]_c \tag{3.11}$$

where $W_{ux}^{'}$ is the cross power spectral density of the white noise, $u(kT)$, and the signal. The subscript $\underline{c}$ denotes the causal part of the transform (terms involving z^{-1}). However

$$W_{ux}^{'}(z) = W_{zx}^{'}(z)\ B(z^{-1}) \tag{3.12}$$

so that

$$g(z) = \left[\frac{W'_{zx}(z)\ B(z^{-1})}{T} \right]_c \tag{3.13}$$

The optimum filter is thus

$$\begin{aligned} H(z) &= G(z)\ B(z) \\ &= \left[\frac{W'_{zx}(z)\ B(z^{-1})}{T} \right]_c \quad B(z) \end{aligned} \tag{3.14}$$

The procedure of taking "the causal part of" is analogous to the spectrum factorization done in Chapter II.

Using the above technique, the previous example is now solved for discrete time signals. The signal and noise correlation functions are

$$\varphi_{xx}(kT) = \frac{1}{2}\ e^{-|k|T} \tag{3.15}$$

$$\varphi_{vv}(kT) = \delta(k) \tag{3.16}$$

The sampled power spectral density of the signal may be obtained by means of Z transforms and is as follows:

$$\begin{aligned} W'_z(x) &= T\ Z \left[\frac{1}{2}\ e^{-|k|T} \right] \\ &= T\ \frac{1 - \epsilon^2}{(1 - \epsilon z^{-1})\ (1 - \epsilon z)} \end{aligned} \tag{3.17}$$

where ϵ equals e^{-T}. For the noise, the sampled power spectral density is

$$W'_v(z) = T \tag{3.18}$$

Therefore, the power spectral density of the signal plus noise input to the filter is

$$\begin{aligned} W'_z(z) &= W'_x(z) + W'_v(z) \\ &= T\ \frac{2 - \epsilon z^{-1} - \epsilon z}{(1 - \epsilon z^{-1})\ (1 - \epsilon z)} \end{aligned} \tag{3.19}$$

This may be written in the following manner for ease in factoring:

$$W_z'(z) = \frac{T\epsilon}{b} \quad \frac{(1 - bz^{-1})\ (1 - bz)}{(1 - \epsilon z^{-1})\ (1 - \epsilon z)} \tag{3.20}$$

where

$$b = \frac{1 - \sqrt{1 - \epsilon^2}}{\epsilon} \tag{3.21}$$

The transfer function needed to produce white noise from the signal and noise input to the filter is then,

$$B(z) = \sqrt{b/\epsilon T} \quad \left[\frac{1 - \epsilon z^{-1}}{1 - bz^{-1}}\right] = \frac{1}{\overline{W}_z(z)} \tag{3.22}$$

Since the signal and the noise are uncorrelated,

$$\varphi_{zx}(k) = \varphi_{xx}(k) \tag{3.23}$$

and

$$W_{zx}'(z) = W_x'(z) \tag{3.24}$$

Thus, from Eq. 3.13,

$$g(z) = \left[\frac{W_x'(z)\ B(z^{-1})}{T}\right]_c$$

$$= \left[\frac{1 - \epsilon^2}{(1 - \epsilon z^{-1})\ (1 - \epsilon z)} \quad \sqrt{b/\epsilon T} \quad \frac{(1 - \epsilon z)}{(1 - bz)}\right]_c \tag{3.25}$$

Simplifying and taking the causal part yields

$$g(z) = \sqrt{b/\epsilon T}\ \frac{1 - \epsilon^2}{1 - b\epsilon} \quad \frac{1}{1 - \epsilon z^{-1}} \tag{3.26}$$

The discrete time transfer function for the optimum filter is then

$$H(z) = g(z)\ B(z)$$

$$= \frac{b}{T\epsilon} \left(\frac{1 - \epsilon^2}{1 - b\epsilon}\right)\left(\frac{1}{1 - bz^{-1}}\right) \tag{3.27}$$

The optimum filter diagram is shown in Figure 9. Since b, ϵ, and T are constants, H(z) may be rewritten as:

$$H(z) = C \frac{1}{1 - bz^{-1}} \tag{3.28}$$

where

$$C = \frac{b(1 - \epsilon^2)}{T\epsilon(1 - b\epsilon)} \tag{3.29}$$

The recursive equation for the filter is developed as follows:

$$H(z) = \frac{\hat{x}(z)}{z(z)} = \frac{C}{1 - bz^{-1}}$$

$$\hat{x}(z)\ (1 - bz^{-1}) = Cz(z)$$

$$\hat{x}(z) = bz^{-1}\ \hat{x}(z) + Cz(z)$$

$$\hat{x}(k + 1) = b\hat{x}(k) + CZ(k + 1) \tag{3.30}$$

The error is

$$E(k) = \hat{X}(k) - X(k) \tag{3.31}$$

For the specific example the coefficients for the filter, Eq. 3.30, become b = .869 and c = 12.45. These differ from the coefficients for the discrete Wiener-Kolmogorov filter, Eq. 2.3-11. One might expect that these results would be identical, but although Schwarz and Friedland[17] consider both developments the discrepancy is not explained and remains a subject for further investigation.

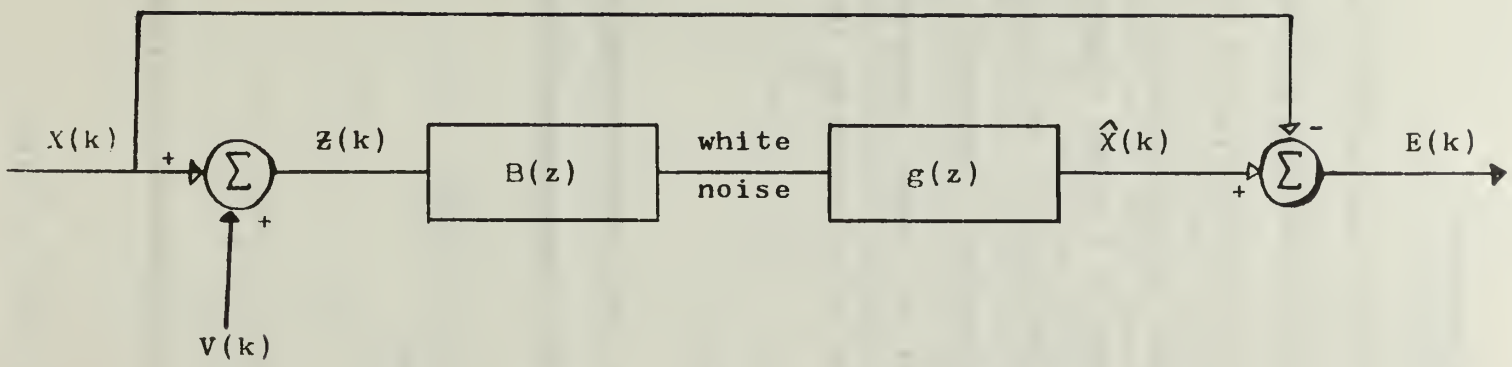


FIGURE 9

DISCRETE BODE-SHANNON FILTER

CHAPTER IV

THE KALMAN FILTER

1. DISCRETE TIME SOLUTION

The objective in this chapter is to examine the filtering problem from the state point of view as developed by Kalman.[9,10,11] In the previous cases, the signal was characterized by either the autocorrelation function or the power spectral density. Here the signal is characterized by the output of a linear dynamic system driven by white noise similar to Bode-Shannon. This presents no problem, for if the power spectral density is rational, it can always be represented as the output of a linear system driven by white noise. As in Chapter II, the linear system has the transfer function,

$$G_1(s) = \frac{1}{s + 1} \tag{4.1-1}$$

Since this is a first order system, the observation matrix is unity and scalar notation can be used. The corresponding differential equation is

$$\frac{dx(t)}{dt} = -x(t) + w(t) \tag{4.1-2}$$

Then, from state variable theory,

$$\Phi = e^{-T} \quad \text{and} \quad \Gamma = 1 - e^{-T}$$

and the differential equation may be expressed in equivalent recursive form:

$$X(k + 1) = e^{-T} X(k) + (1 - e^{-T})W(k) \tag{4.1-3}$$

The signal model is outlined in diagram form in Figure 10; $w(t)$ and $v(t)$ are independent gaussian random processes with zero means and covariances given by:

$$\mathrm{cov}\,[w(t),\ w(\tau)] = Q\delta\ (t - \tau)$$

$$\mathrm{cov}\,[v(t),\ v(\tau)] = R\delta\ (t - \tau)$$

$$\mathrm{cov}\,[w(t),\ v(\tau)] = 0$$

Q and R are the variances of the signal white noise source and the measurement noise, which are set equal to unity in order to duplicate the model for the Wiener filter. The noise sources are simulated in the exact manner as described in Chapter II, Section 4.

The Kalman filter[11] is shown in Figure 11. $G(k)$ is an adjustable gain which minimizes the mean square error,

$$\overline{E^2(n)} = \frac{1}{n} \sum_{k=1}^{n} (E(k))^2 \qquad (4.1\text{-}4)$$

where

$$E(k) = \hat{X}(k) - X(k)$$

$G(k)$ is determined by the following equations:

$$G(k) = \frac{P(k/k - 1)}{P(k/k - 1) + R} \qquad (4.1\text{-}5)$$

$$P(k/k) = (1 - G(k))\ P(k/k - 1) \qquad (4.1\text{-}6)$$

$$P(k + 1/k) = \Phi^2\ P(k/k) + QD \qquad (4.1\text{-}7)$$

where $P(k/k - 1)$ denotes P at time k given the value at time k - 1.

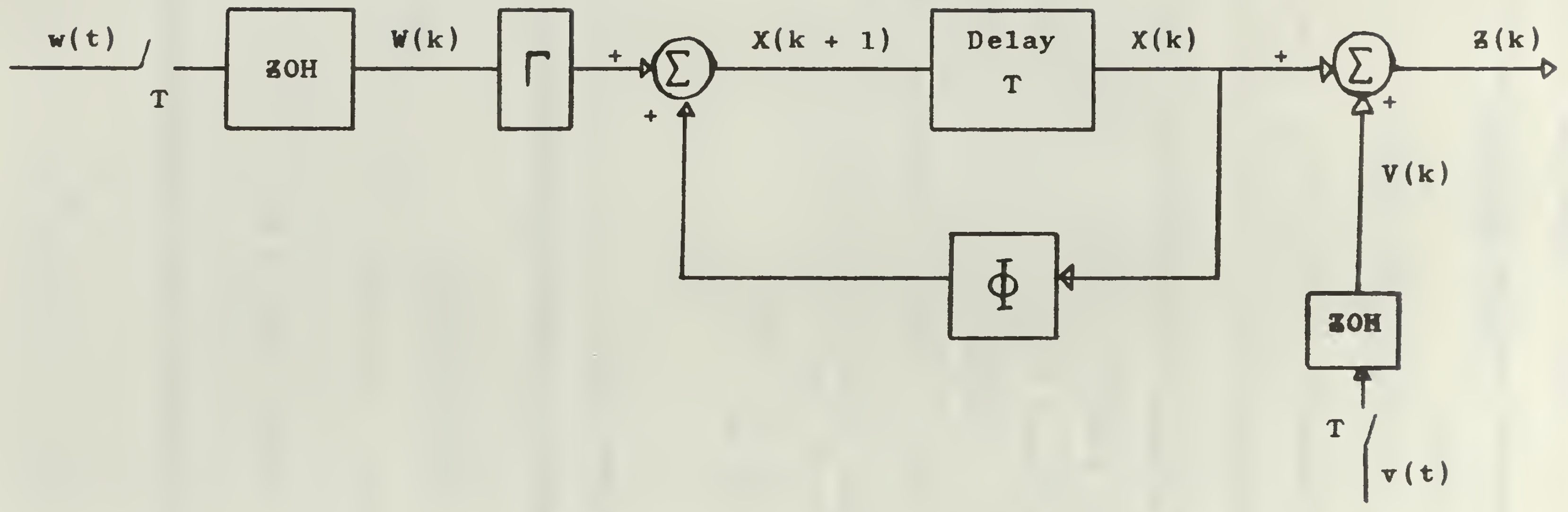


FIGURE 10

KALMAN SIGNAL MODEL

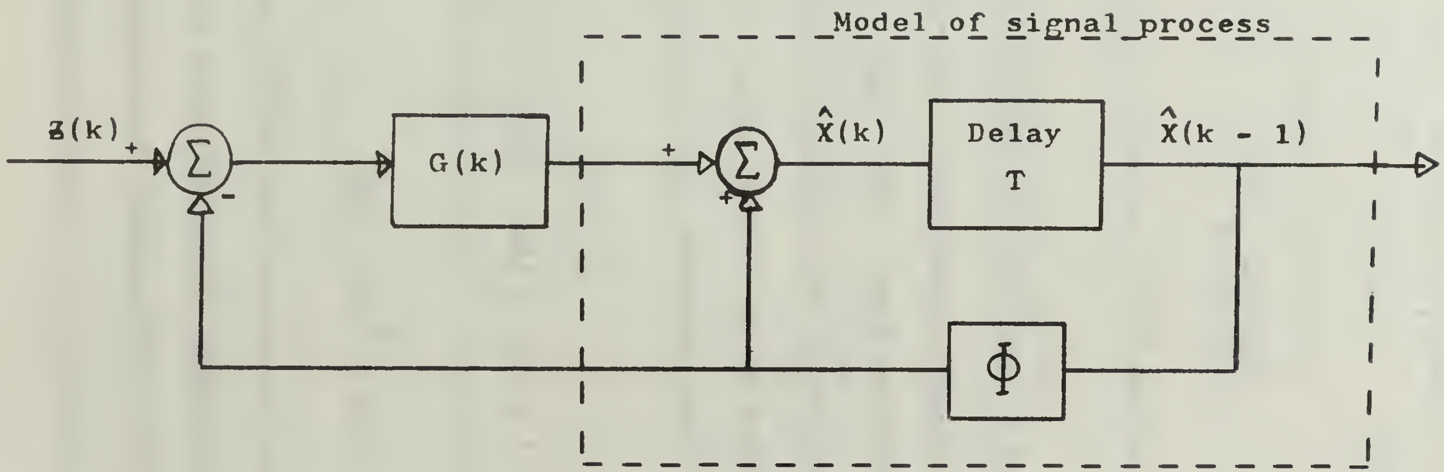


FIGURE 11

DISCRETE KALMAN FILTER

$$P(k/k-1) \triangleq E\left\{\left[Z(k)-Z(k/k-1)\right]^2\right\} = \frac{1}{k}\sum_{n=0}^{k-1}\left[Z(n+1) - Z(n+1/n)\right]^2$$

$$P(k/k) \triangleq E\left\{\left[Z(k) - Z(k/k)\right]^2\right\} = \frac{1}{k}\sum_{n=1}^{k}\left[Z(n) - Z(n/n)\right]^2$$

also,

$$QD = \Gamma^2 Q$$

It is necessary to define P(1/0) in order to start the recursive process.

It is important to note that once P(1/0) is specified the gain equations, Eqs. 4.1-5, 4.1-6, and 4.1-7, are recursive and do not depend on the observations Z(kT). They depend upon the variance of the signal driving noise, the variance of the measurement noise, and the characteristics of the linear system used for signal generation.

From Figure 11, the recursive equation for the Kalman filter may be expressed as

$$\hat{X}(k) = \Phi\hat{X}(k - 1) + G(k)\left[Z(k) - \Phi\hat{X}(k - 1)\right] \qquad (4.1\text{-}8)$$

The error is

$$E(k) = \hat{X}(k) - X(k) \qquad (4.1\text{-}9)$$

As in the Wiener simulation, the sampling interval is 0.01 seconds, $\hat{X}(0)$ is the initial estimate at time t = 0 and the length of the computer run is 3,000 samples. The computer program for the Kalman filter simulation is included as Appendix E.

2. EQUIVALENCE OF KALMAN AND WIENER-KOLMOGOROV FILTERS IN THE STEADY STATE

It has been shown by C. E. Hutchinson[6] that the continuous Kalman and Wiener filters are equivalent in the scalar case after the steady state condition is reached, i.e., G(k) is a constant. This equivalence is developed as follows.

Given a message spectrum, $W_x(\omega) = \dfrac{a^2}{b^2\omega^2 + 1}$

a noise spectrum, $W_v(\omega) = c^2$

and a crosscorrelation function, $\varphi_{xv}(\tau) = 0$

The optimal causal filter, in the Wiener sense, is then

$$H(s) = (k_2 - k_1) \frac{1}{s + k_2} \tag{4.2-1}$$

where

$$k_1 = 1/b$$

$$k_2 = 1/bc \sqrt{a^2 + c^2}$$

Now consider the Kalman filter. The signal model is, in general, defined by:

$$\dot{x}(t) = Fx(t) + w(t) \tag{4.2-2}$$

$$z(t) = Mx(t) + v(t) \tag{4.2-3}$$

The best estimate of x, designated $\hat{x}$, is obtained from the continuous filter equations which correspond to the discrete equations 4.1-5, 4.1-6, 4.1-7, and 4.1-8.

$$\dot{\hat{x}}(t) = F\hat{x}(t) + g(t)\,[z(t) - Mx(t)] \qquad (4.2\text{-}4)$$

where

$$g(t) = P(t)\,M^T R^{-1}$$

$$\dot{P}(t) = FP(t) + Pf^T(t) - P(t)M^T R^{-1} MP(t) + Q$$

and

$$\mathrm{cov}\,[w(t),\, w(\tau)] = Q\delta(t - \tau)$$

$$\mathrm{cov}\,[v(t),\, v(\tau)] = R\delta(t - \tau)$$

$$\mathrm{cov}\,[w(t),\, v(\tau)] = 0$$

If a one dimensional problem is considered with

$$F = -1/b$$

$$Q = a^2/b^2$$

$$R = c^2$$

$$M = 1$$

the Kalman solution becomes

$$\dot{\hat{x}}(t) = -1/b\,\hat{x}(t) + g(t)\,[z(t) - x(t)] \qquad (4.2\text{-}5)$$

where

$$g(t) = P(t)/c^2$$

$$\dot{P}(t) = -\,(2/b)P(t) - (1/c^2)P^2(t) + a^2b^2$$

In the steady state, when P(t) is identically zero, the equation for the Kalman filter reduces to

$$\dot{\hat{x}}(t) = -1/b\,\hat{x}(t) + \left[-1/b + 1/bc\,\sqrt{a^2 + c^2}\,\right]\left[z(t) - \hat{x}(t)\right] \qquad (4.2\text{-}6)$$

The transfer function relating the estimated signal, $\hat{X}(s)$, to the signal plus the noise, $Z(s)$, can be expressed as

$$\frac{\hat{X}(s)}{Z(s)} = (k_2 - k_1) \quad \frac{1}{s + k_2} \tag{4.2-7}$$

which is identical to Eq. 4.2-1. This result can be used as a check on computer programs for the Kalman and Wiener-Kolmogorov filters. When G(k) has reached a constant value, the mean square errors should be identical.

CHAPTER V

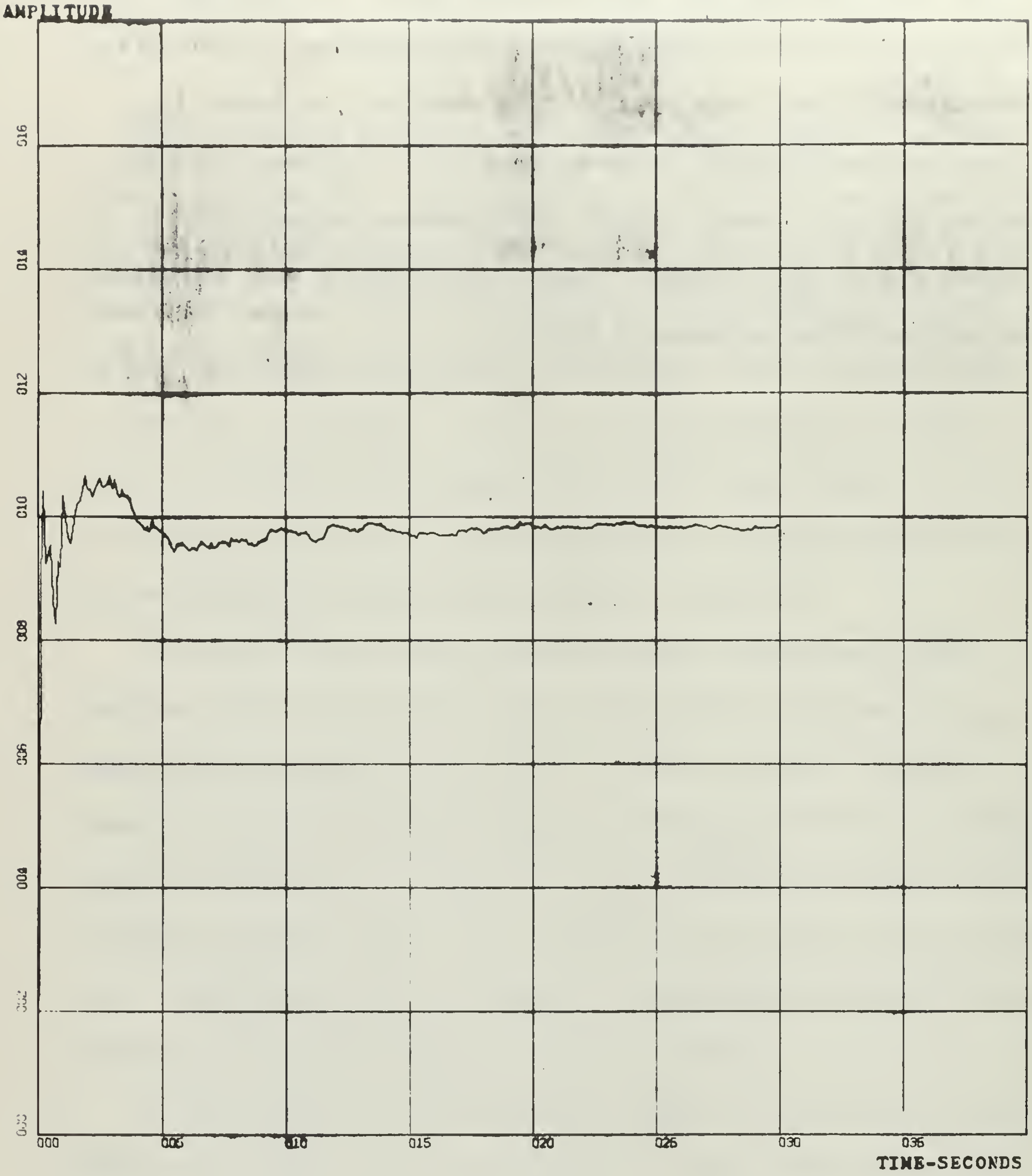
RESULTS AND COMPARISONS

The continuous Kalman and Wiener-Kolmogorov filters have been shown, in Chapter IV, to be equivalent in the steady state condition, that is, when the Kalman gain is constant. This result is verified experimentally, as shown by the computer curves of Figures 16, 20, 24, and 28. The steady state gain of the Kalman filter reaches a value of .0041335 as expected. As soon as the gain reaches its steady value, the curves of mean square error versus time (Figures 18, 19, 22, 23, 26, 27, 30, and 32) are identical to each other and to the Wiener-Kolmogorov filter curves (Figures 14 and 15).

In order to start the recursive process, the Kalman filter requires a priori knowledge of the initial error covariance, $P_{1/0}$. Four different values of $P_{1/0}$ (0.0, .414065, 1.0, and 10.0) were chosen to compare the effectiveness of the Kalman filter for different values of this term. The value of $P_{1/0} = .415065$ was chosen because it gives the steady state gain (.0041335) at the start of the computer run. With this $P_{1/0}$ the performance of the Kalman and Wiener-Kolmogorov filters are identical (Figures 13-15 and 21-23).

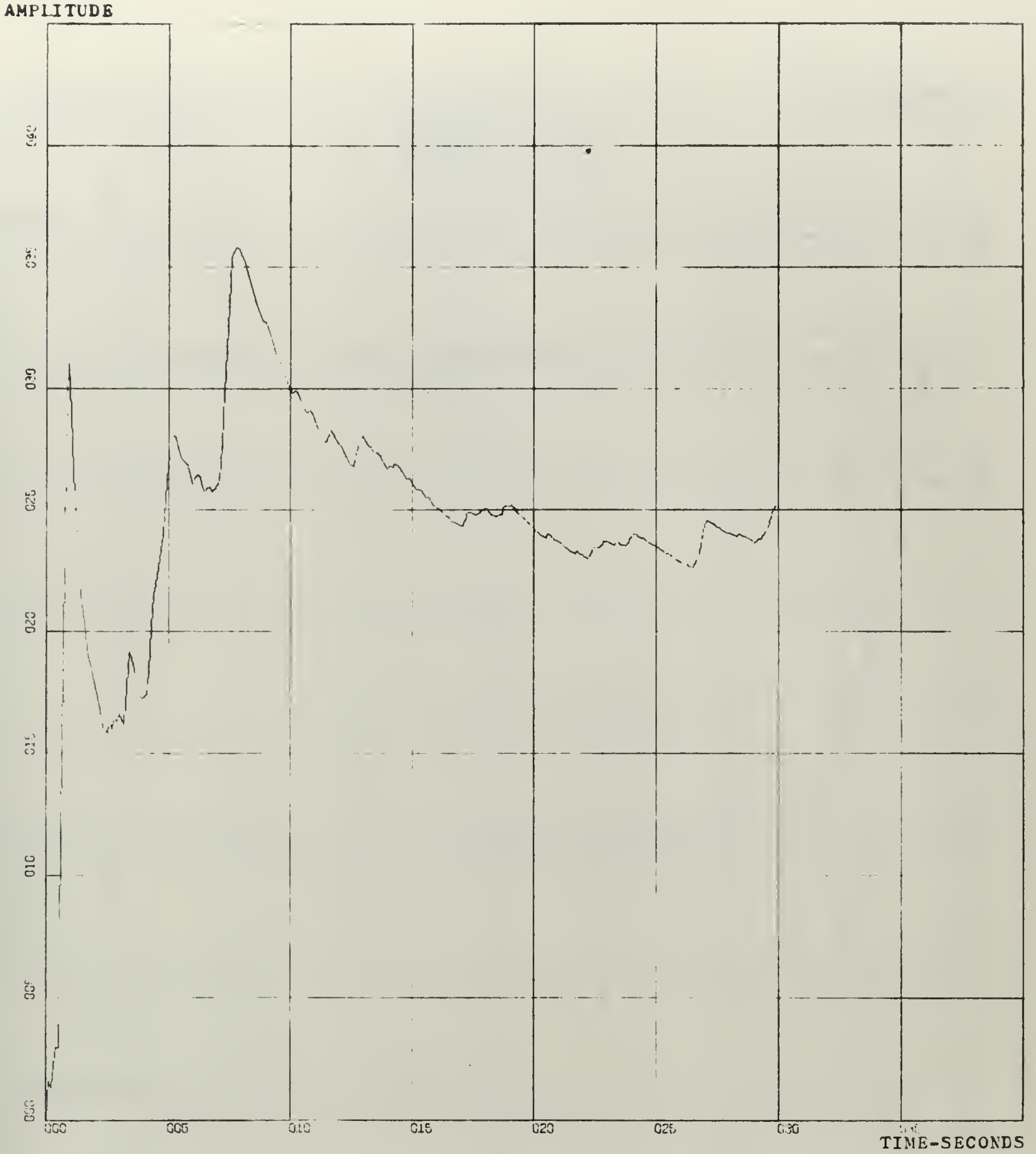
In simulating the non-stationary signal, two different initial values were chosen; one to correspond to a small initial error in estimation and the other to correspond to a large error in the initial estimation. In a tracking problem, the first case would represent the situation where a target is picked up at long range. The second case represents the situation where a target suddenly appears at relatively short range. For small initial errors, the Wiener-Kolmogorov

and the Kalman filters performed essentially the same (Figures 13, 14, 17, 18, 21, 22, 25, 26, 29, and 30). For large initial errors the Kalman filter, with large initial covariance of error, $P_{1/0}$, gives the best transient response (Fig. 31). For large initial error, if $P_{1/0}$ is equal to zero, the transient response of the Kalman filter takes somewhat longer time to settle than the Wiener-Kolmogorov filter as shown in Fig. 19.



X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=2.00E+01 UNITS INCH.
NO FILTER M.S.E. VS TIME
FLETCHER H.G. THESIS

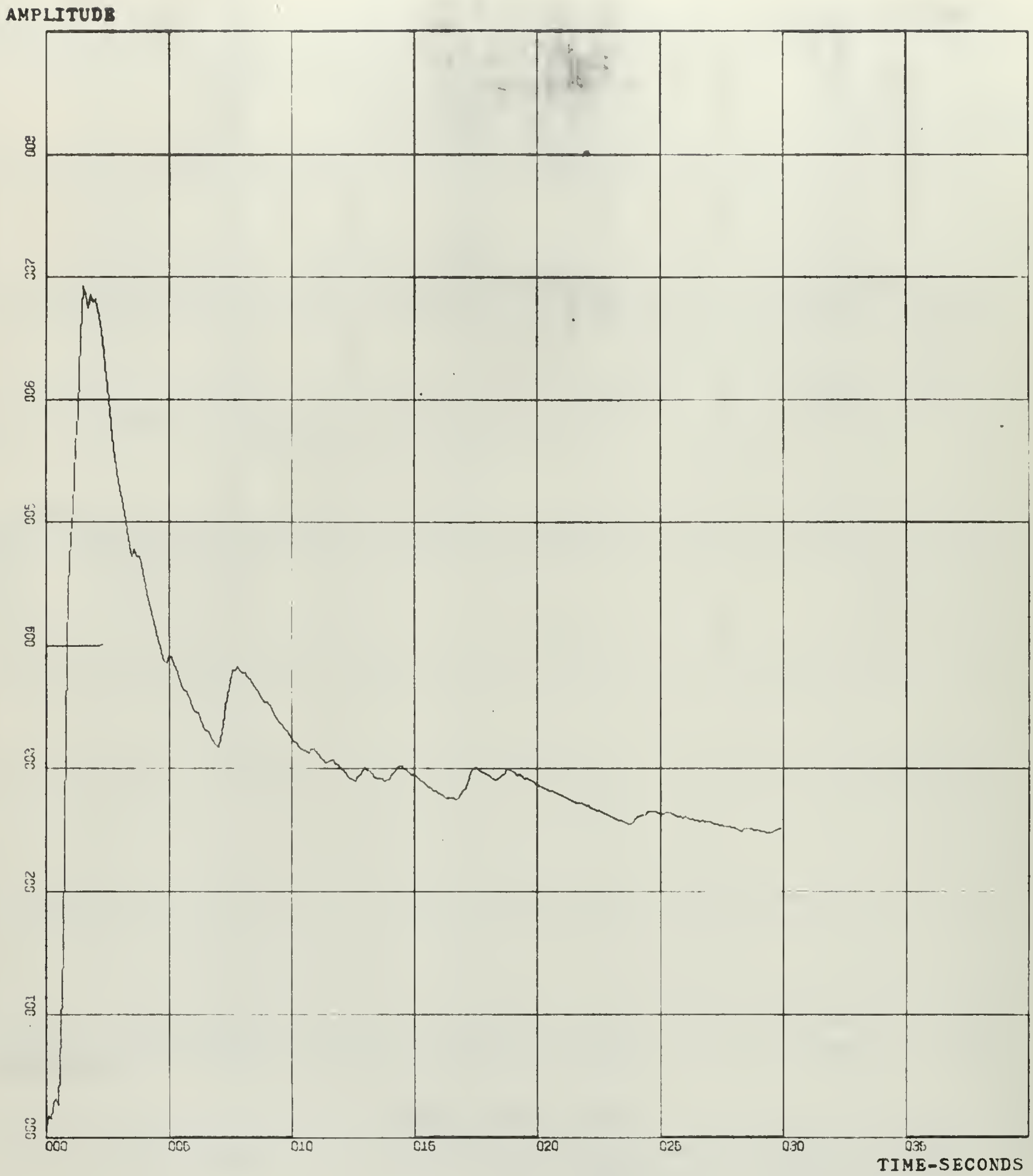
FIGURE 12



SMALL INITIAL ERROR

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-02 UNITS INCH.
WIENER FILTER VARIANCE OF E. VS TIME
FLETCHER H.G. THESIS

FIGURE 13

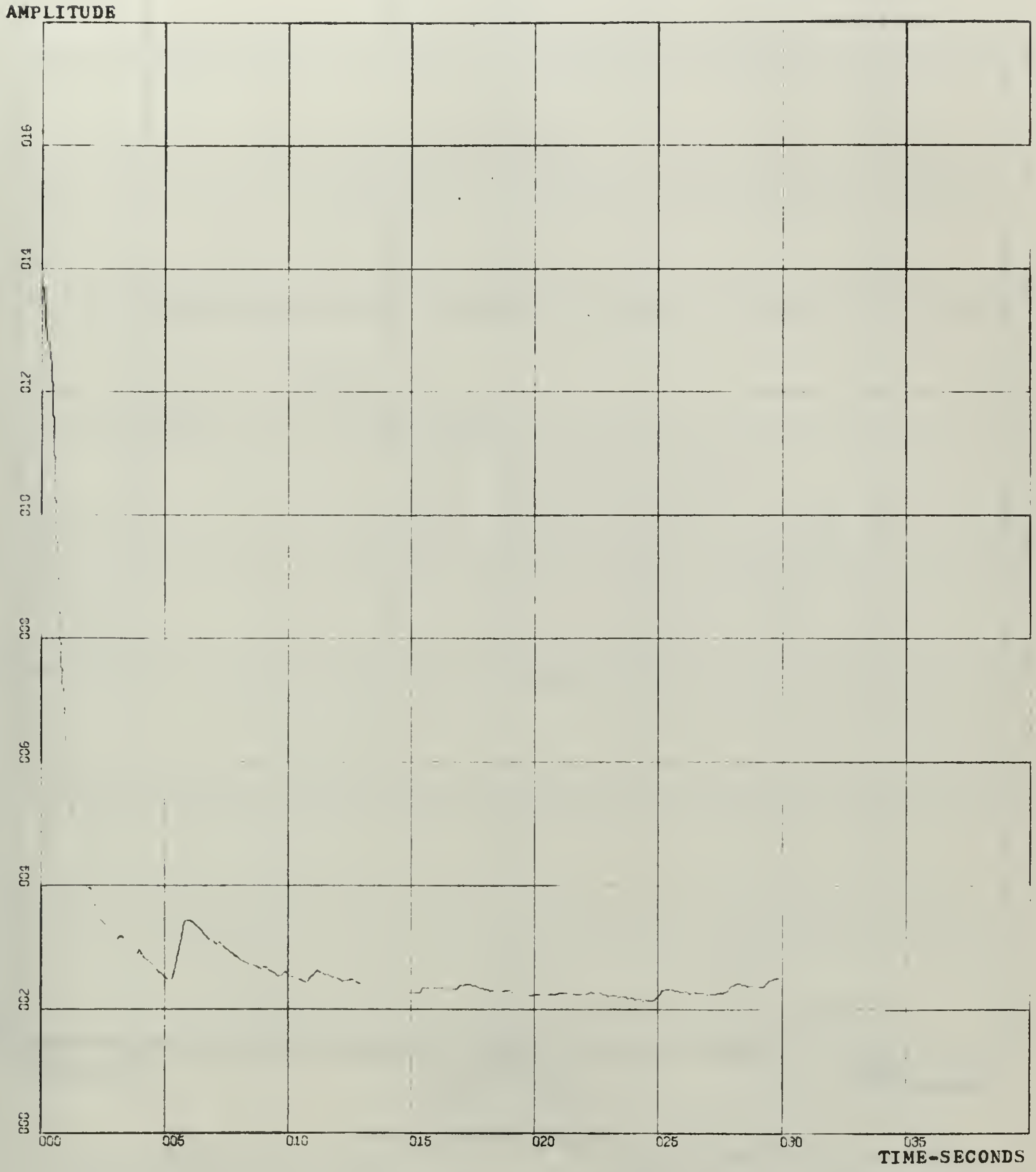


SMALL INITIAL ERROR

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=1.00E-01 UNITS INCH.

WIENER FILTER M.S.E. VS TIME
FLETCHER H.G. THESIS

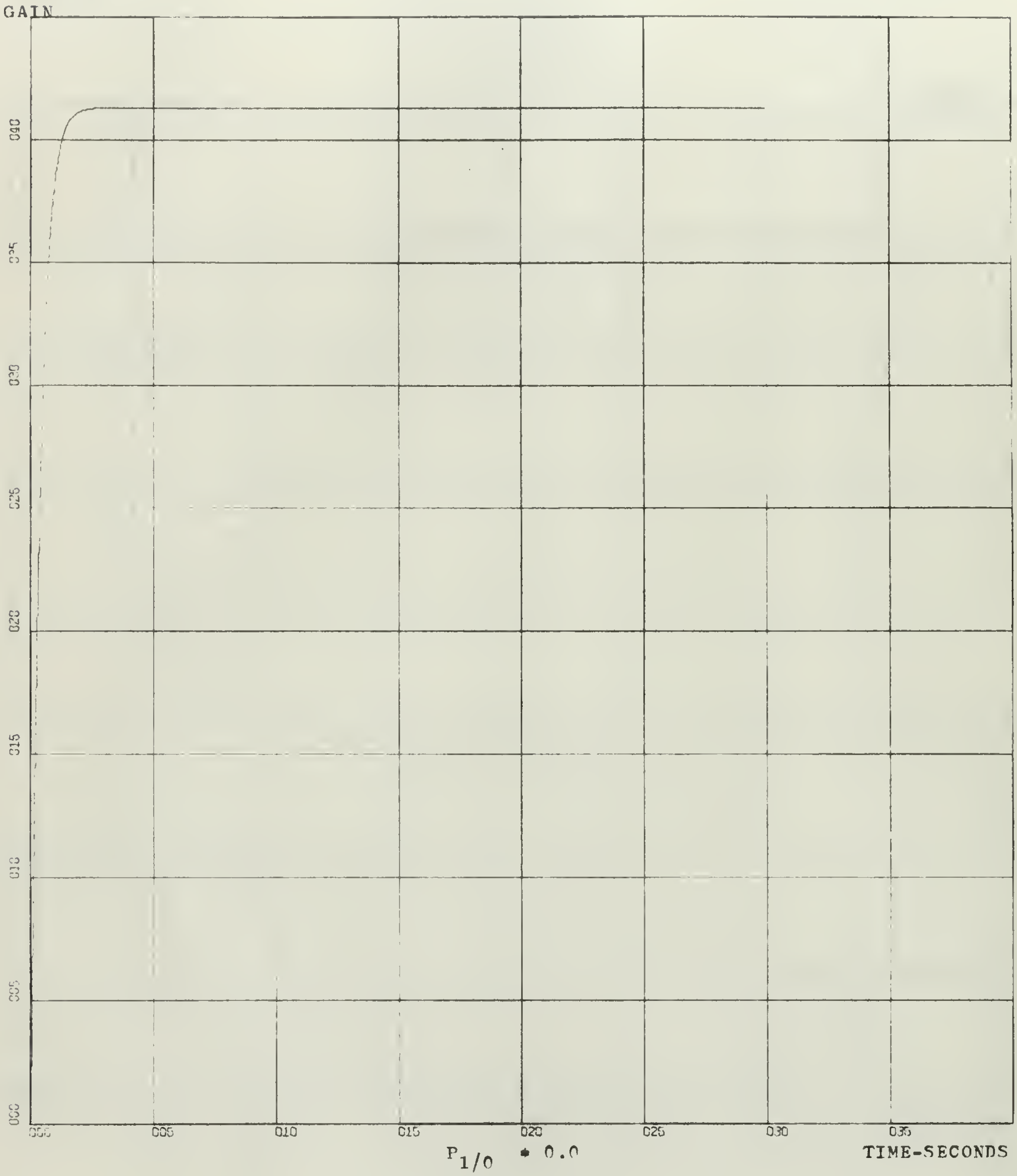
FIGURE 14



LARGE INITIAL ERROR

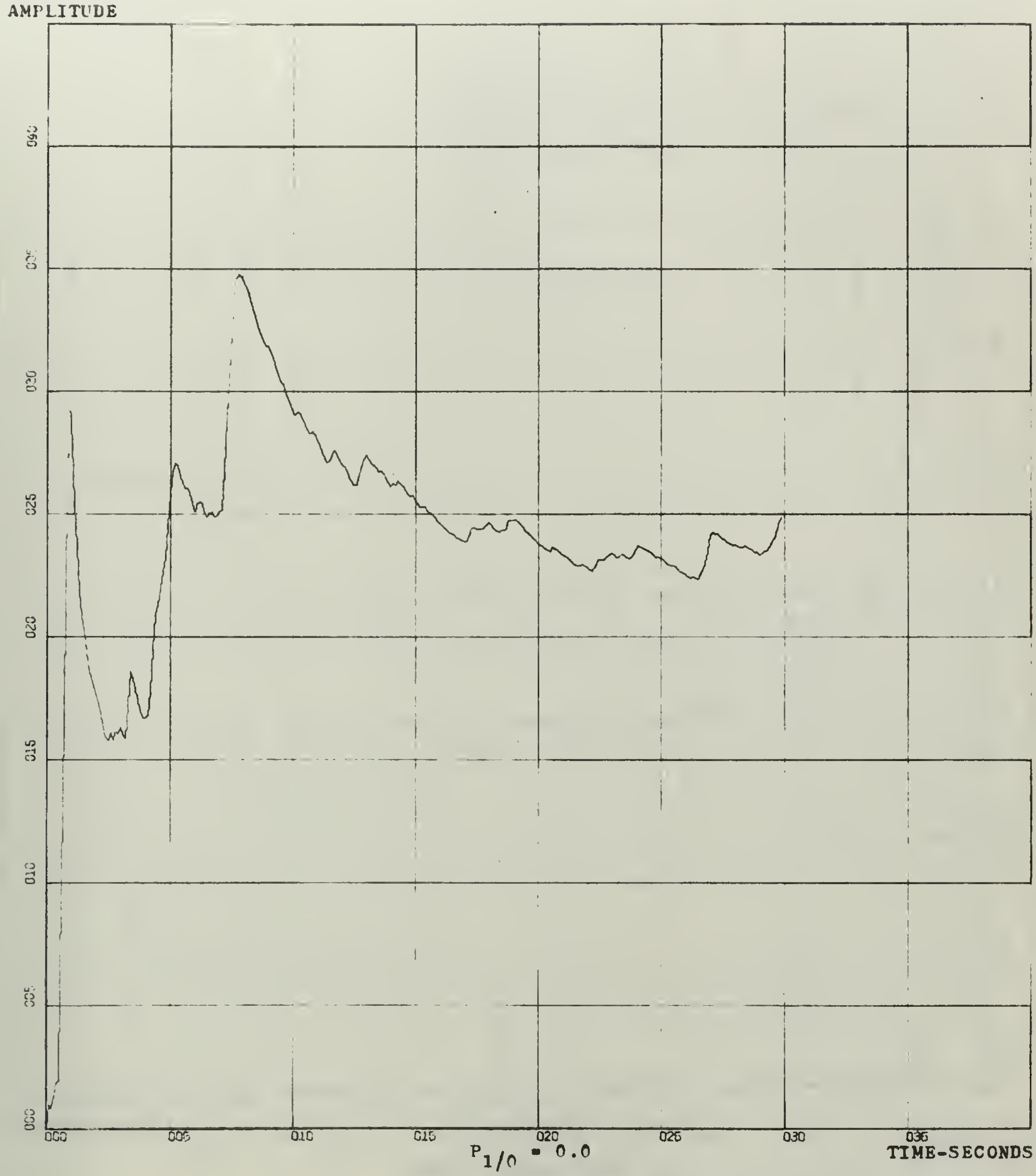
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=2.00E-01 UNITS INCH.
WIENER FILTER M.S.E. VS TIME
FLETCHER H.G. THESIS

FIGURE 15



X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-04 UNITS INCH.
KALMAN FILTER GAIN VS. TIME
FLETCHER H.G. THESIS 1

FIGURE 16



SMALL INITIAL ERROR

X-SCALE=5.00E+01 UNITS INCH.

Y-SCALE=5.00E-02 UNITS INCH.

KALMAN FILTER VARIANCE OF E. VS TIME

FLETCHER H.G. THESIS 1

FIGURE 17

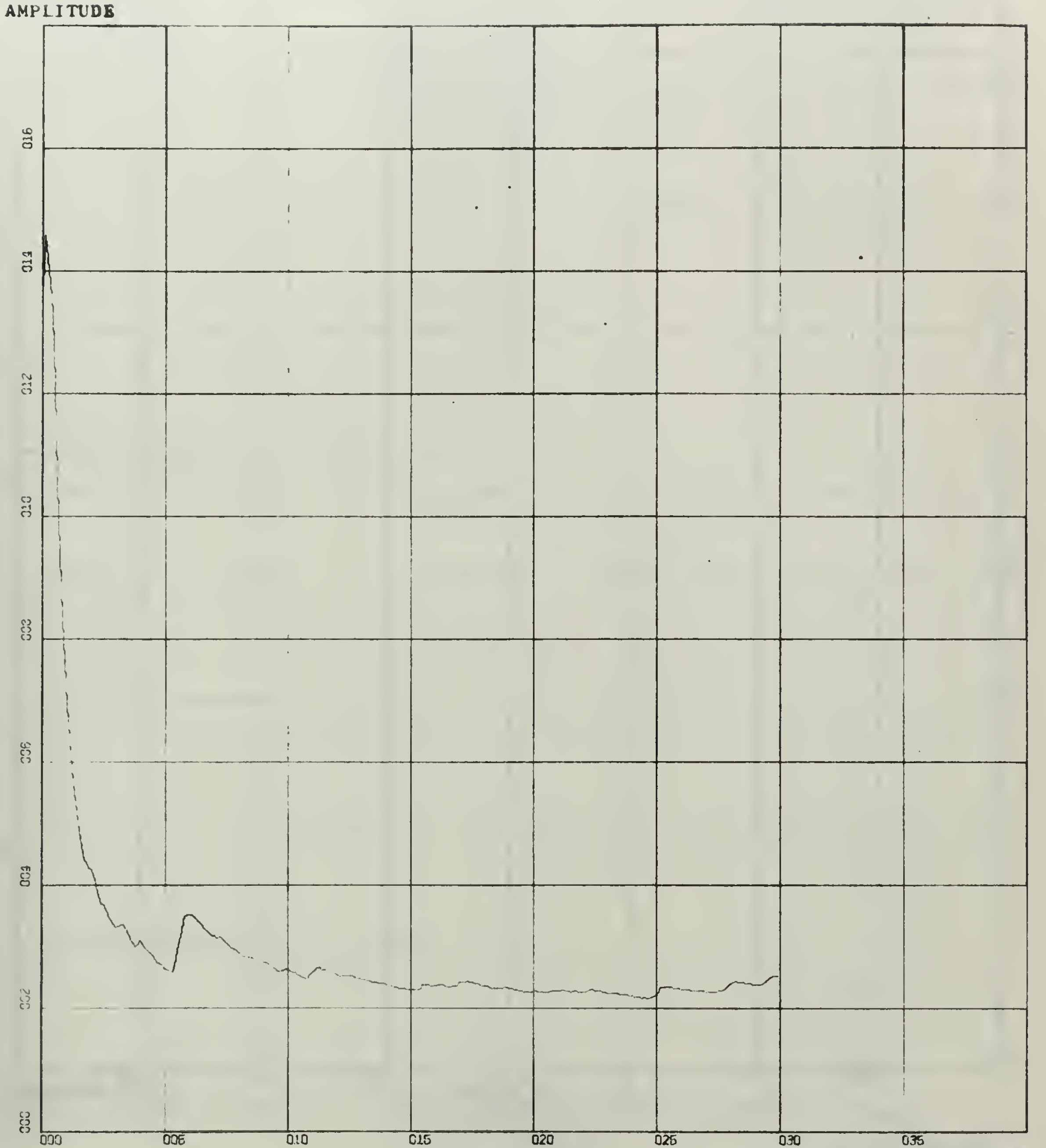
$P_{1/0} = 0.0$

SMALL INITIAL ERROR

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=1.00E-01 UNITS INCH.

KALMAN FILTER M.S.E. VS TIME
FLETCHER H.G. THESIS 1

FIGURE 18



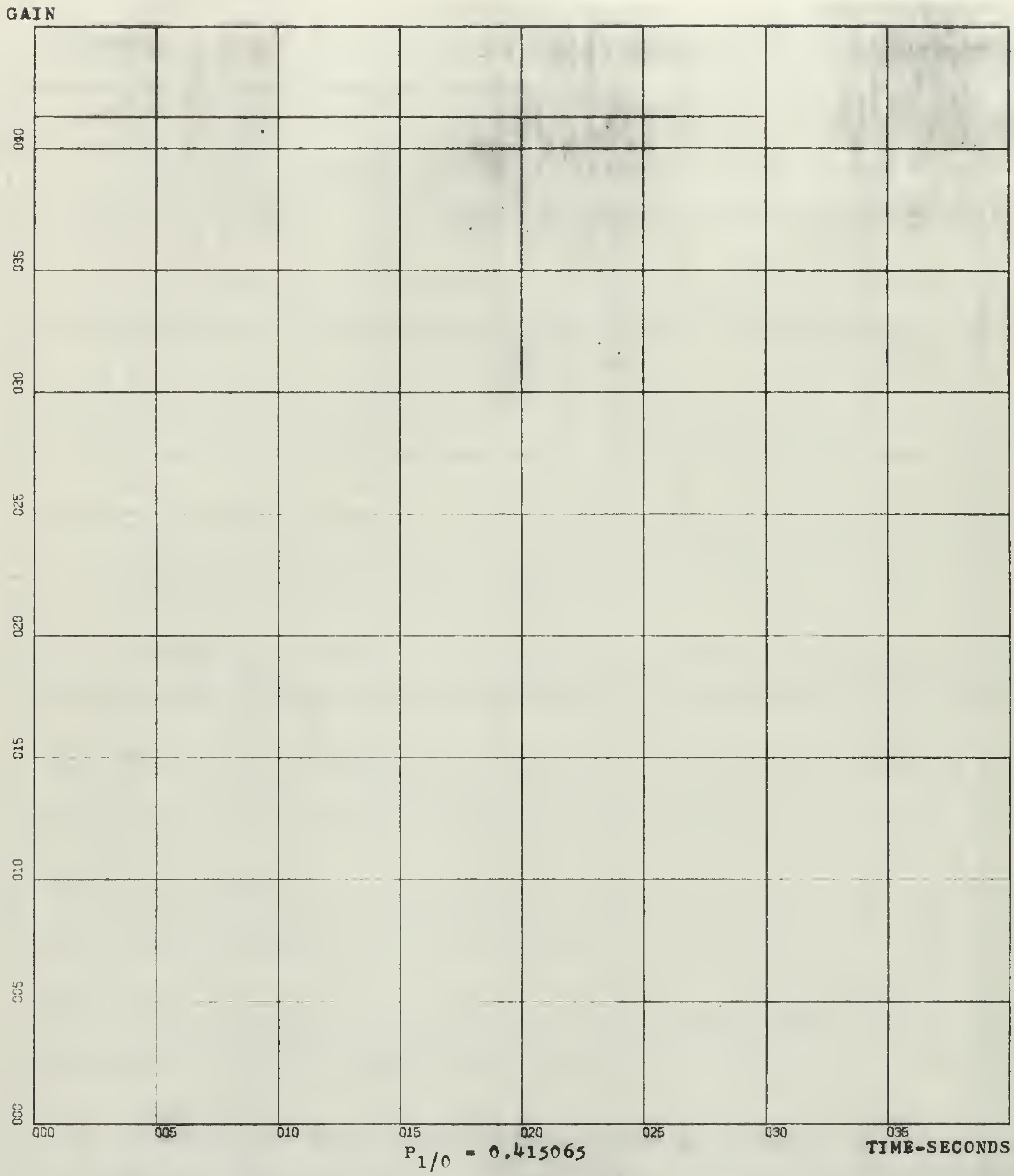
$P_{1/0} = 0.0$

LARGE INITIAL ERROR

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=2.00E-01 UNITS INCH.

KALMAN FILTER M.S.E. VS TIME
FLETCHER H.G. THESIS 1

FIGURE 19



$P_{1/0}$ = 0.415065

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-04 UNITS INCH.

KALMAN FILTER GAIN VS. TIME
FLETCHER H.G. THESIS 1

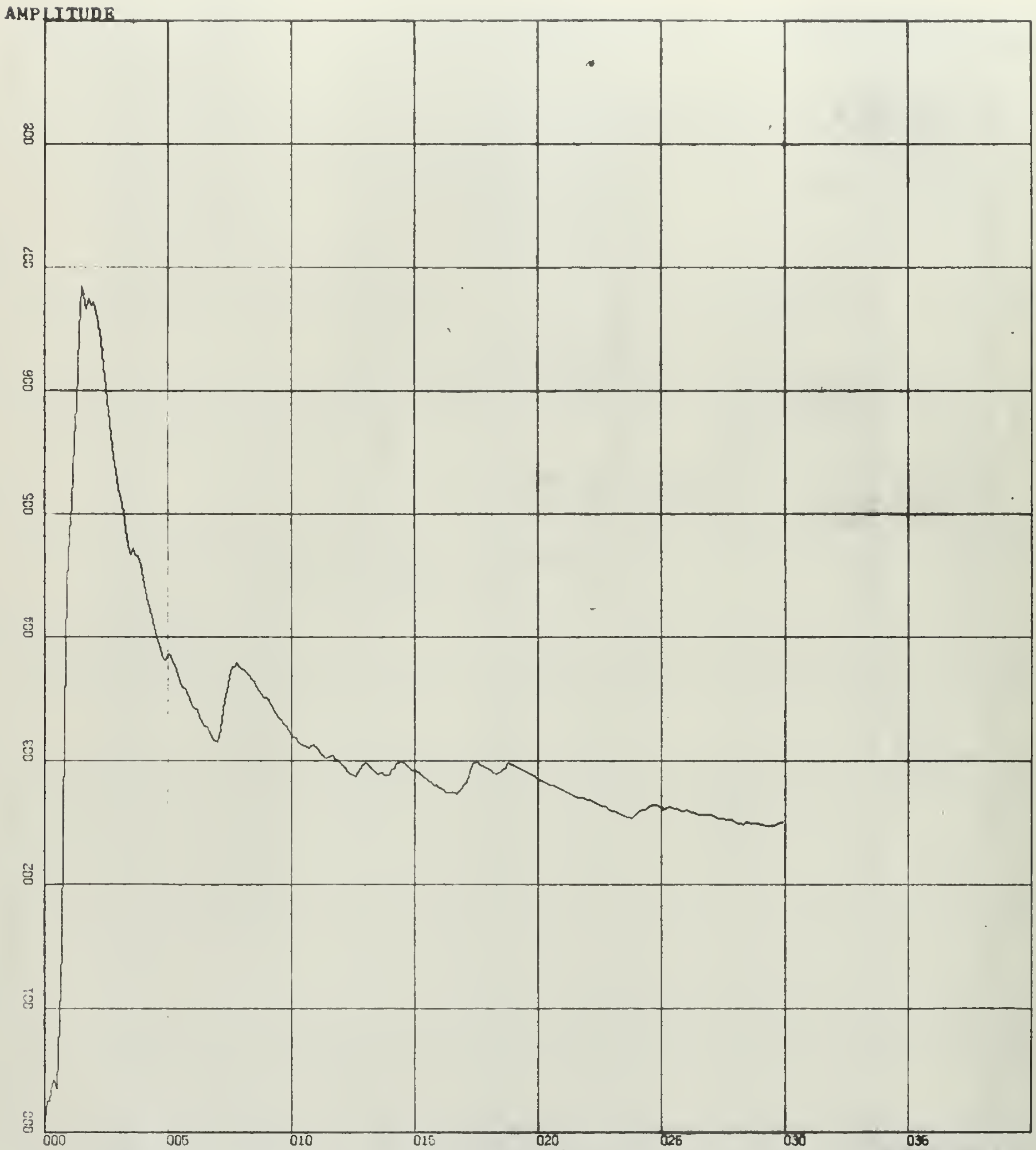
FIGURE 20

$P_{1/0} = 0.415065$

SMALL INITIAL ERROR

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-02 UNITS INCH.
KALMAN FILTER VARIANCE OF E. VS TIME
FLETCHER H.G. THESIS 2

FIGURE 21



$P_{1/0} = 0.415065$

SMALL INITIAL ERROR

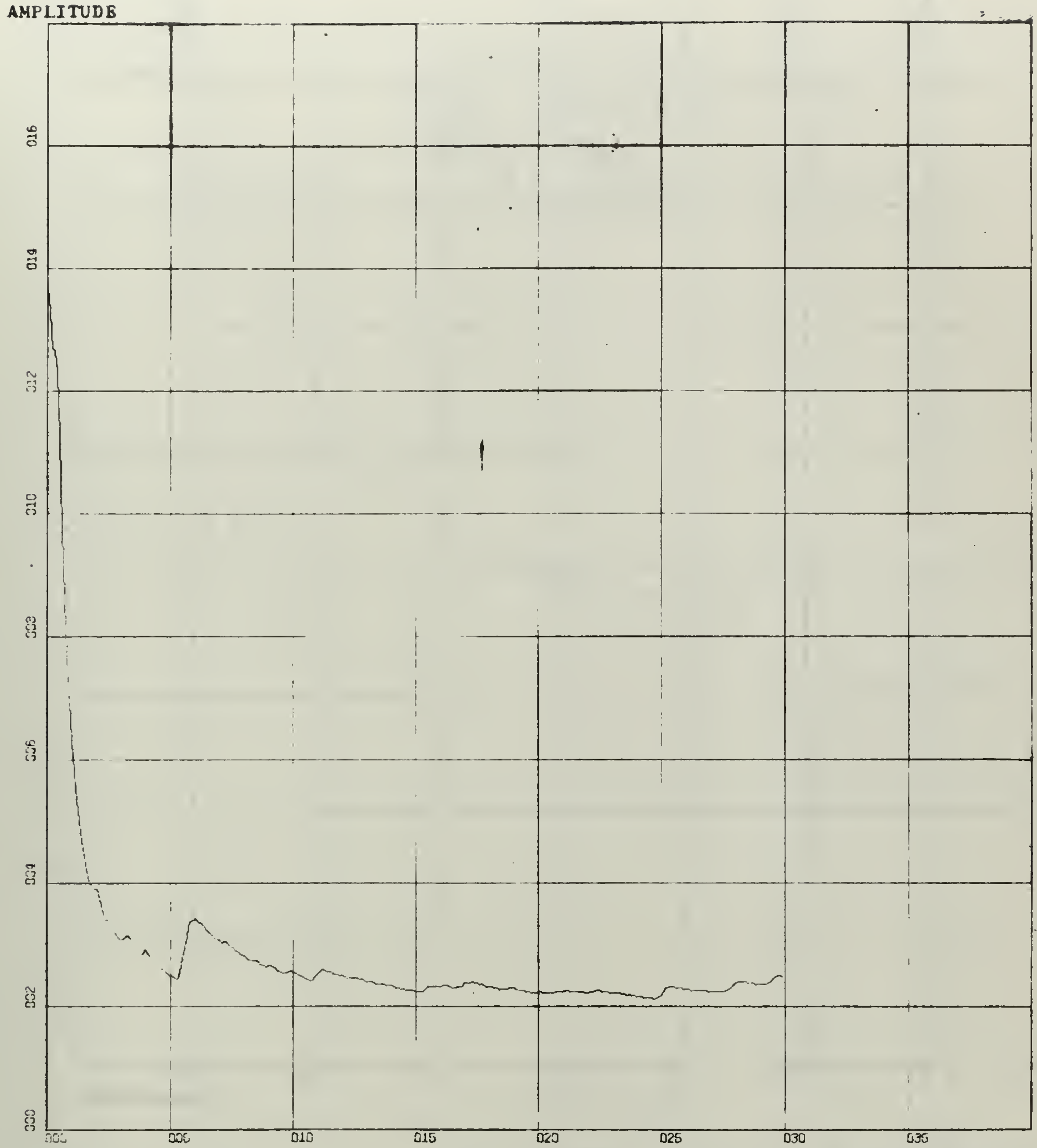
X-SCALE=5.00E+01 UNITS INCH.

Y-SCALE=1.00E-01 UNITS INCH.

KALMAN FILTER M.S.E. VS TIME

FLETCHER H.G. THESIS 2

FIGURE 22



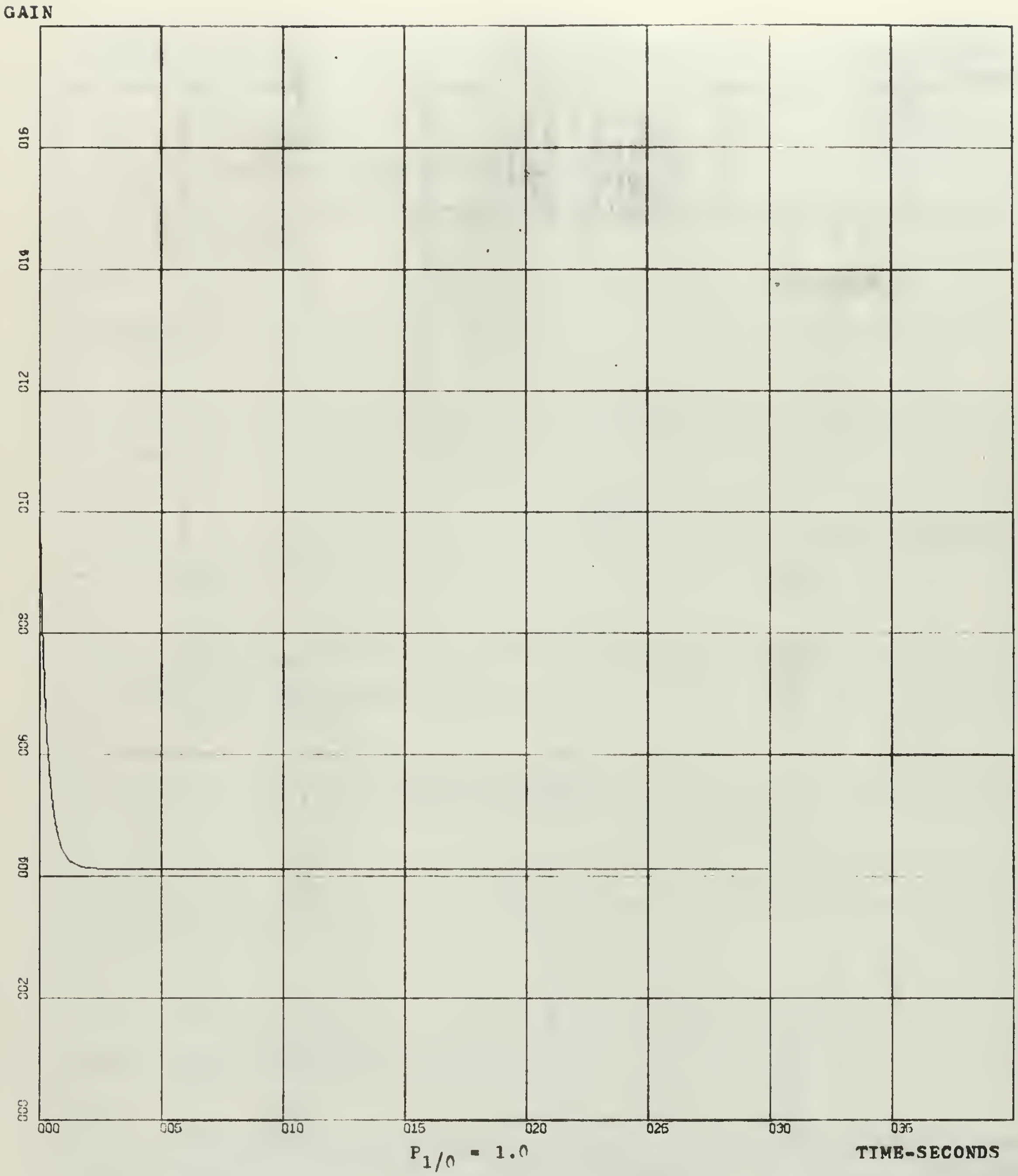
$P_{1/0}$ = 0.415065 TIME-SECONDS

LARGE INITIAL ERROR

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=2.00E-01 UNITS INCH.

KALMAN FILTER M.S.E. VS TIME
FLETCHER H.G. THESIS 2

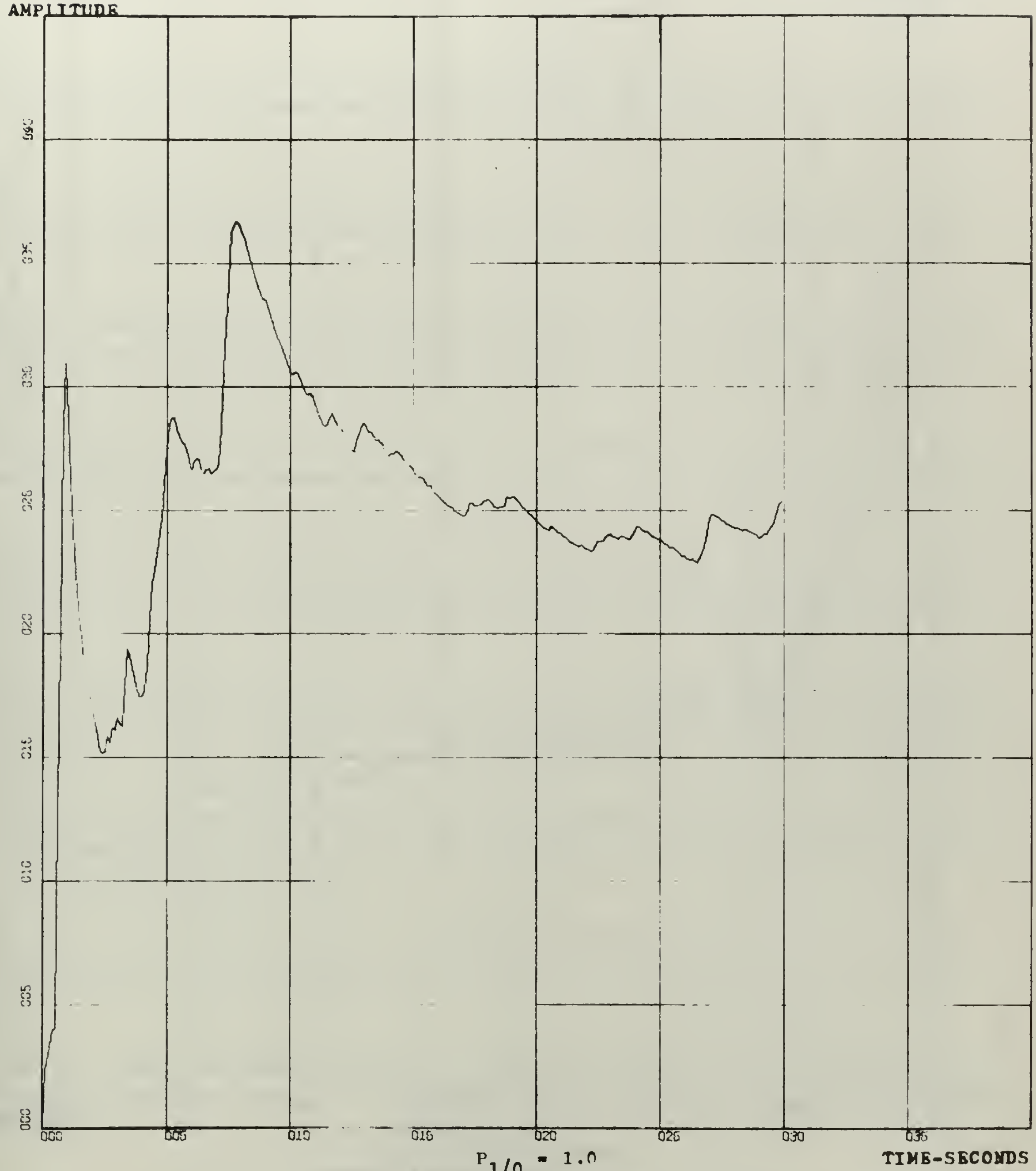
FIGURE 23



X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=2.00E-03 UNITS INCH.

KALMAN FILTER GAIN VS. TIME
FLETCHER H.G. THESIS

FIGURE 24



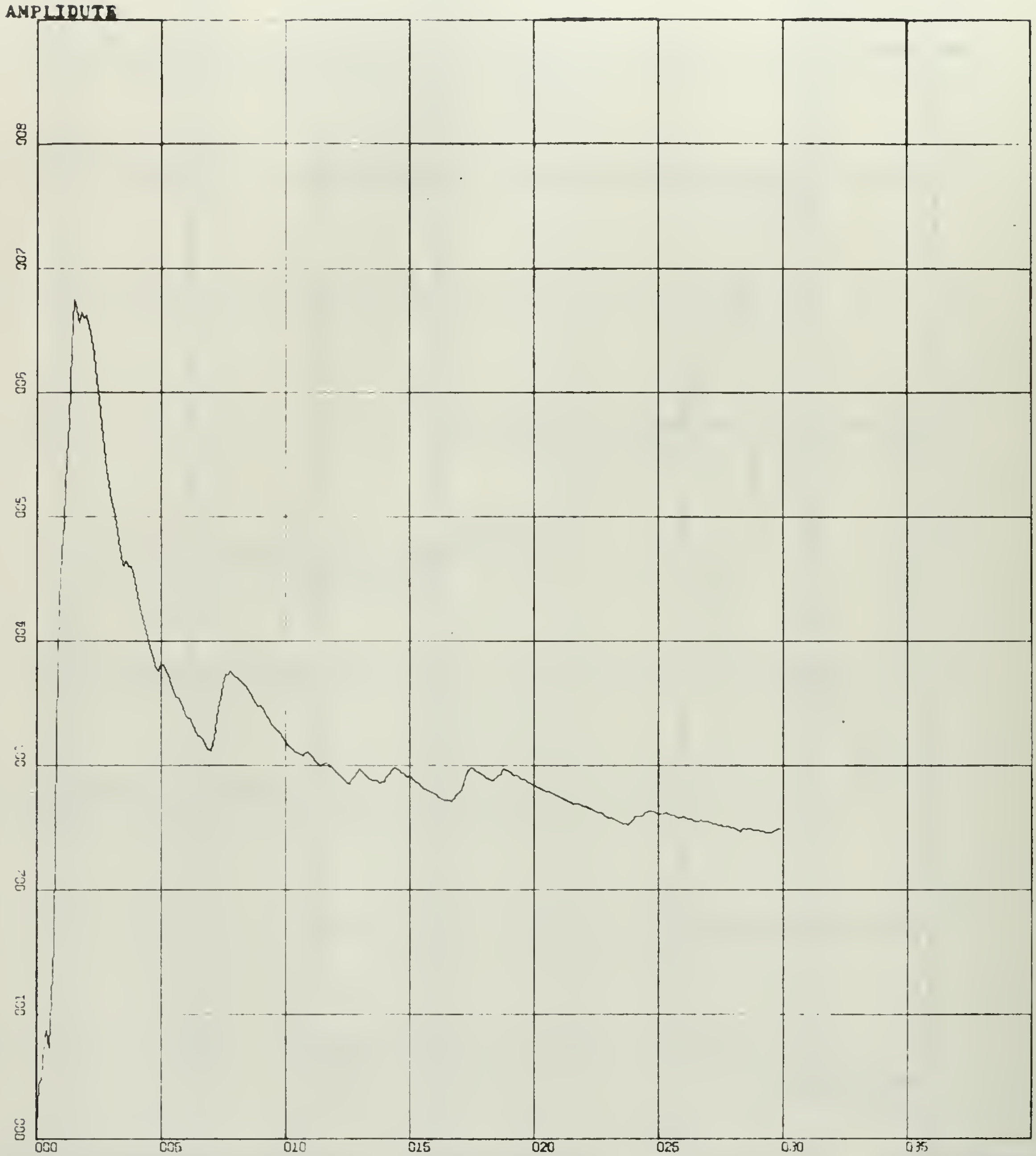
$P_{1/0}$ = 1.0

SMALL INITIAL ERROR

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-02 UNITS INCH.

KALMAN FILTER VARIANCE OF E. VS TIME
FLETCHER H.G. THESIS

FIGURE 26



$P_{1/0} = 1.0$

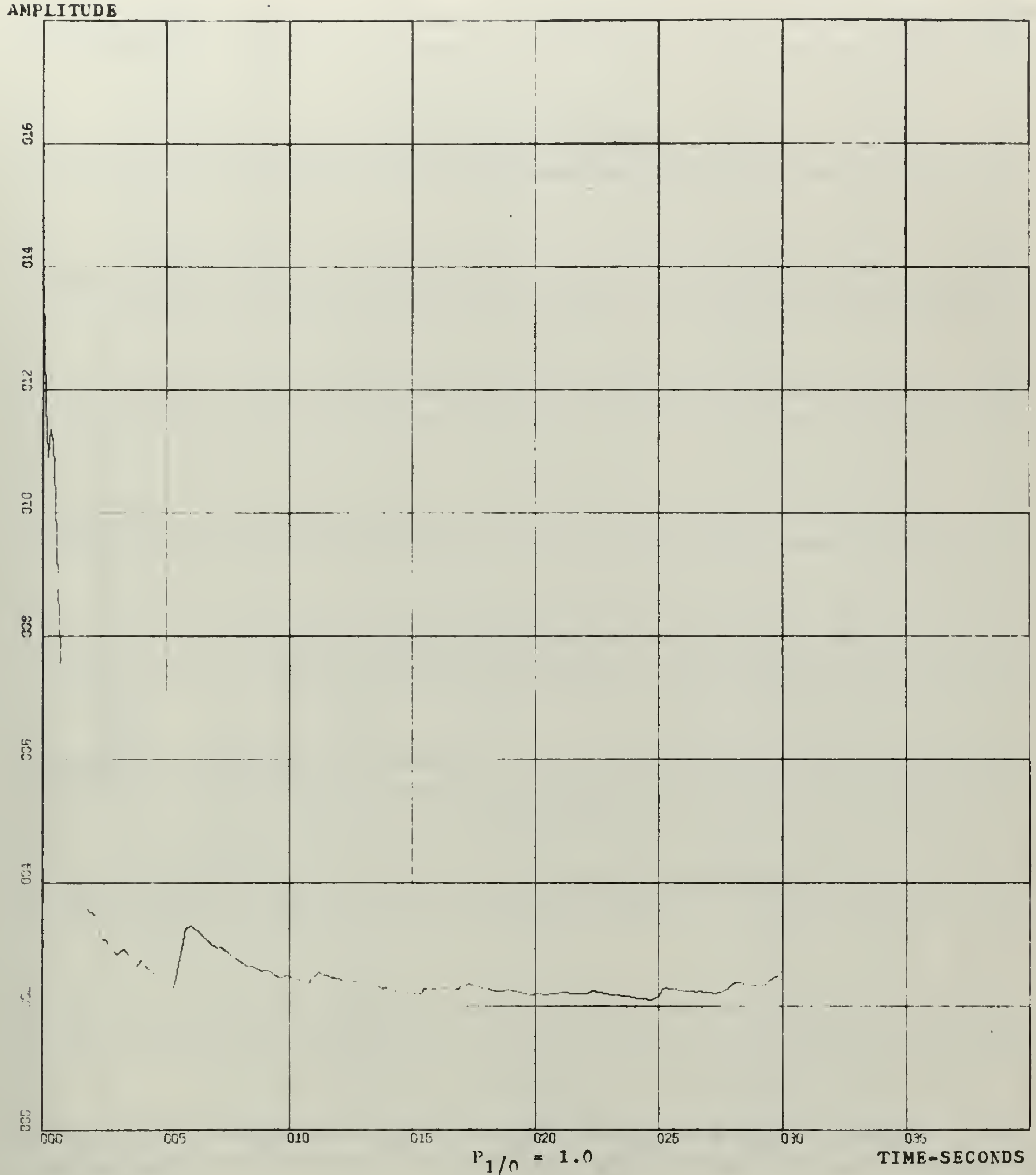
SMALL INITIAL ERROR

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=1.00E-01 UNITS INCH.

KALMAN FILTER M.S.E. VS TIME

FLETCHER H.G. THESIS

FIGURE 26

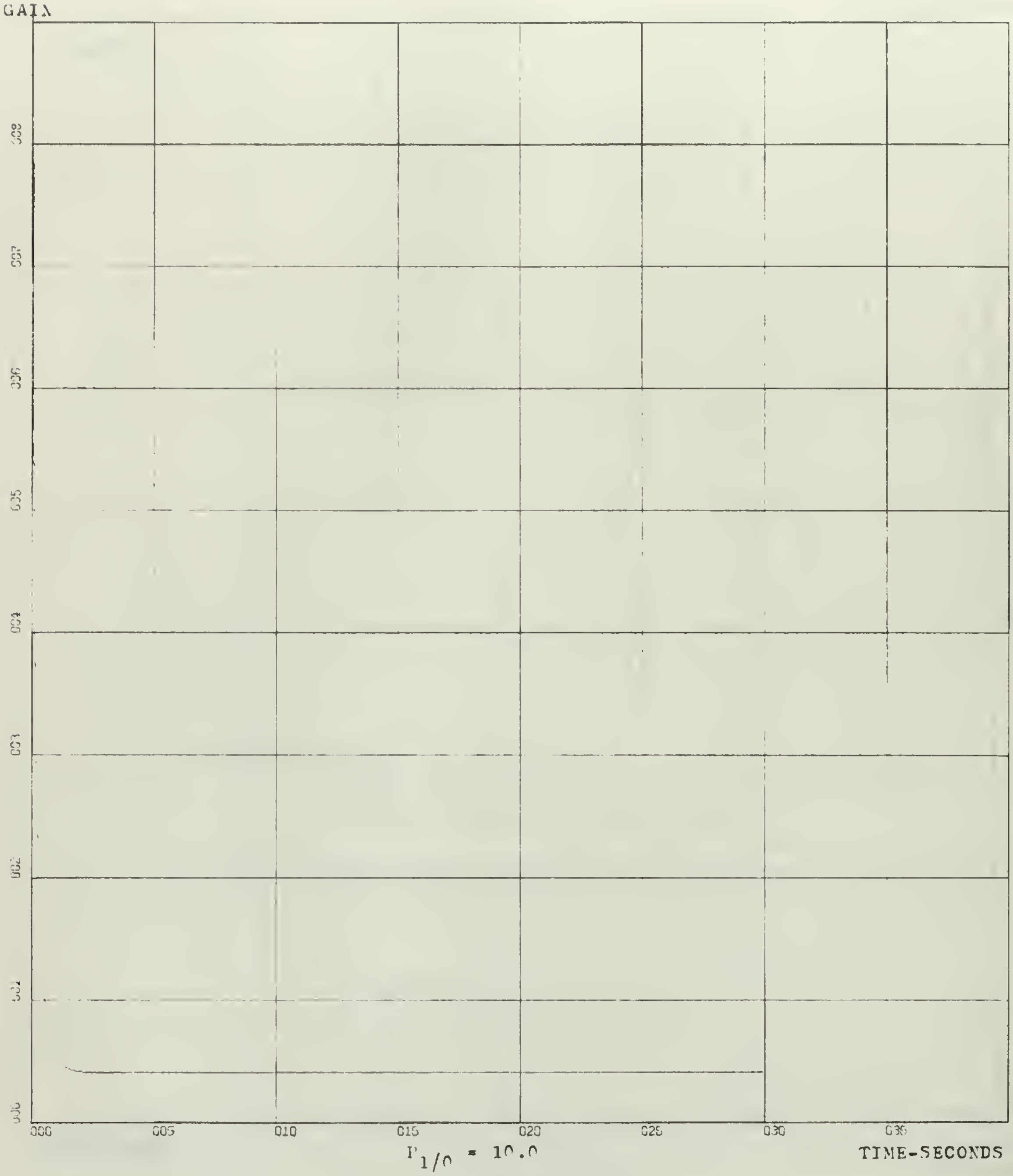


LARGE INITIAL ERROR

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=2.00E-01 UNITS INCH.

KALMAN FILTER M.S.E. VS TIME
FLETCHER H.G. THESIS

FIGURE 27

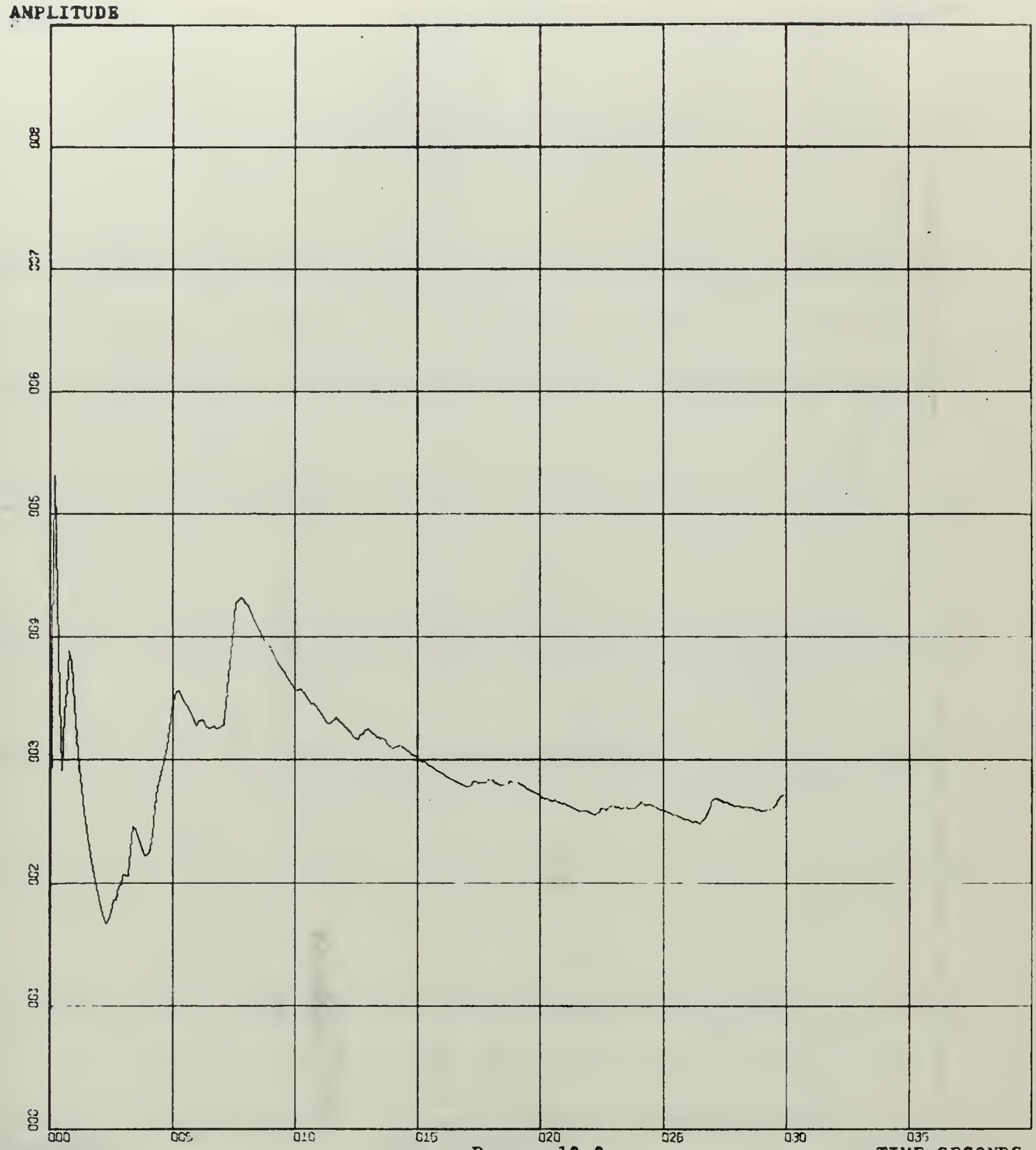


X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=1.00E-02 UNITS INCH.

KALMAN FILTER GAIN VS. TIME

FLETCHER H.G. THESIS 1

FIGURE 28



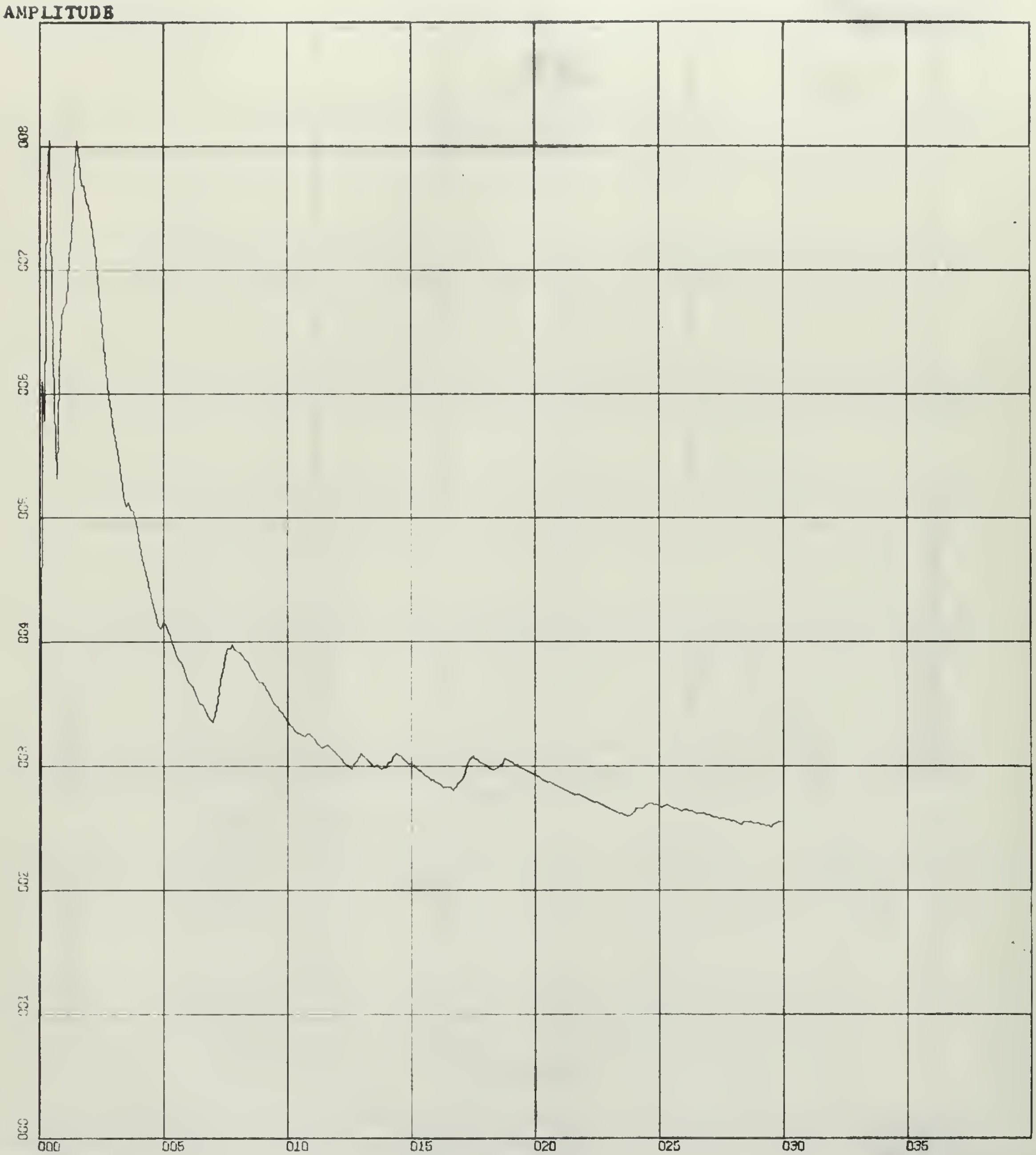
$P_{1/0} = 10.0$

SMALL INITIAL ERROR

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=1.00E-01 UNITS SCALE=1.00E-01 UNITS INCH.

KALMAN FILTER VARIANCE OF E. VS TIME
FLETCHER H.G. THESIS 3

FIGURE 29

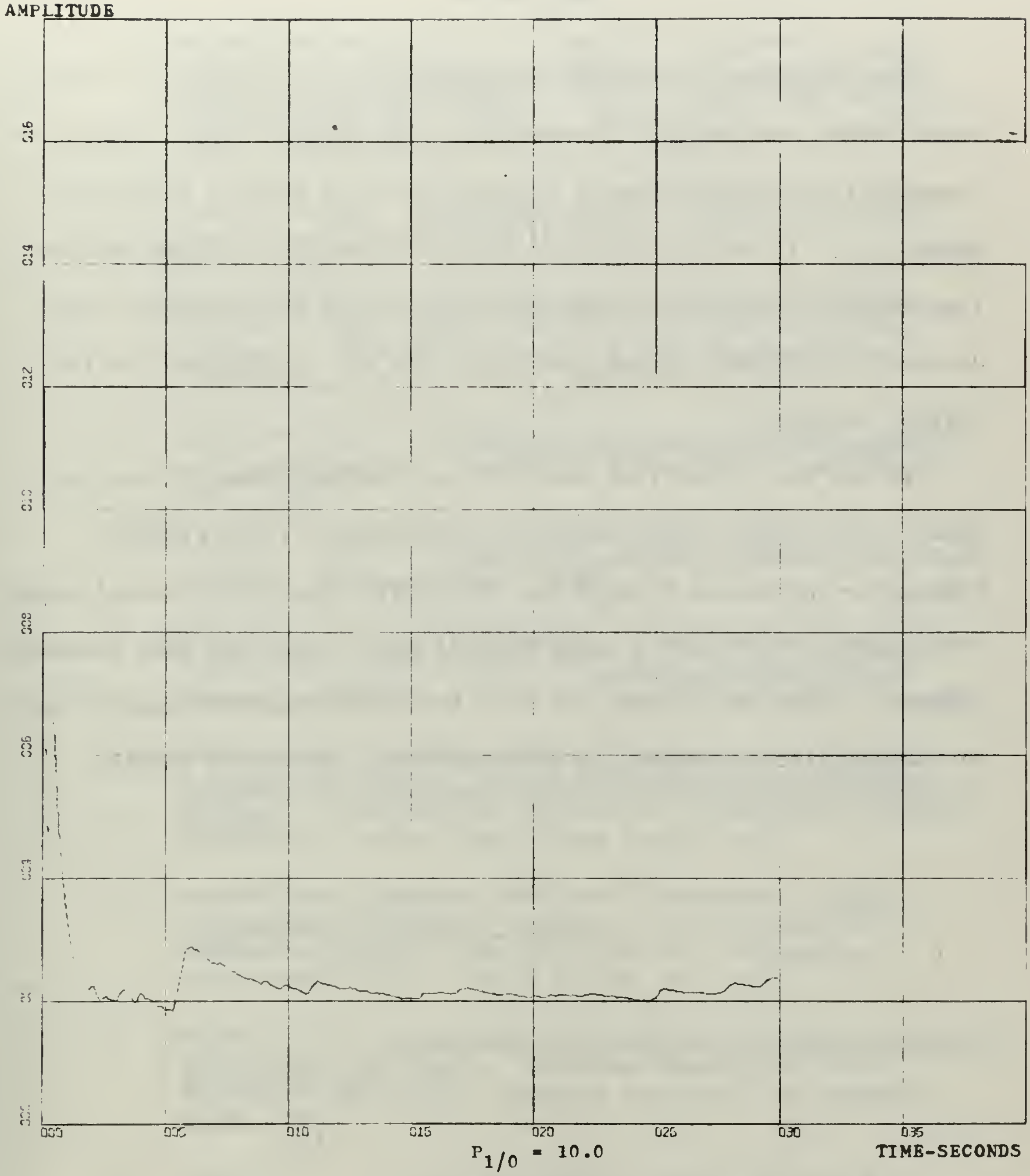


$P_{1/0}$ = 10.0

SMALL INITIAL ERROR

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=1.00E-01 UNITS INCH.
KALMAN FILTER M.S.E. VS TIME
FLETCHER H.G. THESIS 3

FIGURE 30



LARGE INITIAL ERROR

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=2.00E-01 UNITS INCH.

KALMAN FILTER M.S.E. VS TIME
FLETCHER H.G. THESIS 3

FIGURE 31

CHAPTER VI

CONCLUSIONS

The responses of the discrete Kalman filter for various initial error covariance and the discrete Wiener-Kolmogorov filter have been compared for a non-stationary random input with defined statistical properties. It was necessary to develop a computer program for the simulation of the desired spectrum from a white noise source. The discrete time Wiener-Kolmogorov filter was also programmed for the digital computer.

It has been shown that when the gain of the Kalman filter is equal to its steady state value the performance of the Wiener-Kolmogorov and Kalman filters are identical. For large initial error the Kalman filter (with a large initial gain) shows the best transient response. When the initial error is small the Wiener-Kolmogorov and the Kalman filters respond in as essentially equivalent fashion.

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APPENDIX A

DEVELOPMENT OF THE WIENER-HOPF INTEGRAL EQUATION

The Wiener-Hopf equation gives the relationship between the filter weighting function, the autocorrelation function of the input to the filter, and the crosscorelation function of the filter input with the desired filter output. The problem is described in Figure 1; $x(t)$ is some signal, $v(t)$ is white noise, $\hat{x}(t)$ is the estimate of the signal, and $e(t)$ is the error in estimation; reduce $e(t)$ to a minimum in the mean squared sense by a correct choice of $h(t)$, the impulse response of the filter. The error by definition is:

$$e(t) \triangleq x(t) - \hat{x}(t) \tag{A.1}$$

Squaring both sides yields,

$$e^2(t) = x^2(t) - 2\hat{x}(t)\, x(t) + \hat{x}^2(t) \tag{A.2}$$

The output, $x(t)$, and the impulse response of the filter, $H(t)$, may be related by means of the convolution integral.

$$\hat{x}(t) = \int_{-\infty}^{\infty} H(\tau)\, x(t - \tau)\, d\tau \tag{A.3}$$

where τ is the variable of integration. Squaring Eq. A.3 yields

$$\hat{x}^2(t) = \left[\int_{-\infty}^{\infty} H(\tau)\, z(t - \tau) d\tau\right] \left[\int_{-\infty}^{\infty} H(\tau_1)\, z(t - \tau_1) d\tau_1\right] \tag{A.4}$$

Substituting Eq. A.4 into Eq. A.2 gives,

$$e^2(t) = x^2(t) - 2\left[\int_{-\infty}^{\infty} H(\tau)\, z(t - \tau) d\tau\, x(t)\right] +$$

$$\left[\int_{-\infty}^{\infty} H(\tau)\, z(t - \tau) d\tau\right] \left[H(\tau_1)\, z(t - \tau_1) d\tau_1\right] \tag{A.5}$$

By definition, the mean square error is,

$$\overline{e^2(t)} \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^2(t)dt \tag{A.6}$$

Substituting Eq. A.5 into Eq. A.6 yields:

$$\overline{e^2(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} x^2(t)dt - 2\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \int_{-\infty}^{\infty} H(\tau)z(t - \tau)x(t)d\tau$$

$$+ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \left[\int_{-\infty}^{\infty} H(\tau)z(t - \tau)dt \int_{-\infty}^{\infty} H(\tau_1)z(t - \tau_1)d\tau_1 \right] \tag{A.7}$$

The definition of the input autocorrelation function is

$$\varphi_{zz}(\tau) \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} z(t)z(t + \tau)dt \tag{A.8}$$

Similarly, the crosscorrelation function between the input and the output is

$$\varphi_{z\hat{x}}(\tau) \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} z(t)\hat{x}(t + \tau)dt \tag{A.9}$$

and the autocorrelation function of the output is,

$$\varphi_{\hat{x}\hat{x}}(\tau) \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \hat{x}(t)\hat{x}(t + \tau)dt \tag{A.10}$$

Now, making use of the correlation functions, and interchanging the order of integration in Eq. A.7 yields,

$$\overline{e^2(t)} = \varphi_{xx}(0) - 2 \int_{-\infty}^{\infty} H(\tau)\, \varphi_{zx}(\tau)d\tau \quad +$$

$$\int_{-\infty}^{\infty} H(\tau)d\tau \int_{-\infty}^{\infty} H(\tau_1)\, \varphi_{zz}(\tau - \tau_1)d\tau_1 \tag{A.11}$$

To determine the system function that minimizes the mean square error, assume that a solution exists and denote this solution as $H_m(t)$. Now construct a weighting function:

$$H(t) = H_m(t) + \delta H_\delta(t) \tag{A.12}$$

where $H_m(t)$ is the assumed solution, $H_\delta(t)$ is any arbitrary realizable weighting function, and δ is a parameter which may be varied to test whether $H_m(t)$ is the solution. The construction is such that at δ equal to zero, the derivative of mean square error with respect to δ must be equal to zero. By setting this derivative equal to zero for δ equal to zero, a condition for $H_m(t)$ which must be satisfied in order for it to be a solution,is specified.

Substituting H(t) as given by Eq. A.12 into Eq. A.11 and differentiating with respect to δ yields:

$$\begin{aligned}\frac{d\left[\overline{e^2(t)}\right]}{d\delta} &= -2\int_{-\infty}^{\infty} H_\delta(\tau)\varphi_{zx}(\tau)d\tau + \int_{-\infty}^{\infty} H_m(\tau)d\tau \int_{-\infty}^{\infty} H_\delta(\tau_1)\varphi_{zz}(\tau - \tau_1)d\tau_1 \\ &+ \int_{-\infty}^{\infty} H_\delta(\tau)d\tau \int_{-\infty}^{\infty} H_m(\tau_1)\,\varphi_{zz}(\tau - \tau_1)d\tau_1 \\ &+ 2\delta\int_{-\infty}^{\infty} H_\delta(\tau)d\tau \int_{-\infty}^{\infty} H_\delta(\tau)\,\varphi_{zz}(\tau - \tau_1)d\tau_1\end{aligned} \tag{A.13}$$

Because of the even property of autocorrelation functions of stationary signals,

$$\varphi_{zz}(\tau_1 - \tau) = \varphi_{zz}(\tau - \tau_1) \tag{A.14}$$

which leads to

$$\int_{-\infty}^{\infty} H_m(\tau)d\tau \int_{-\infty}^{\infty} H_\delta(\tau_1)\, \varphi_{zz}(\tau - \tau_1)d\tau_1$$

$$= \int_{-\infty}^{\infty} H_\delta(\tau)d\tau \int_{-\infty}^{\infty} H_m(\tau_1)\varphi_{zz}(\tau - \tau_1)d\tau_1 \qquad (A.15)$$

Substituting Eq. A.15 in Eq. A.13 and setting

$$\frac{d\left[\overline{e^2(t)}\right]}{d\delta} = 0 \qquad \text{at } \delta = 0$$

yields:

$$2\int_{-\infty}^{\infty} H_\delta(\tau)d\tau\left[\int_{-\infty}^{\infty} H_m(\tau_1)\varphi_{zz}(\tau - \tau_1)d\tau_1 - \varphi_{zx}(\tau)\right] = 0 \qquad (A.16)$$

Since $H_\delta(\tau)$ is a realizable weighting function, it must be zero for values of t less than zero. The only way that Eq. A.16 can be satisfied for $\tau \geq 0$ is that the factor in the brackets be equal to zero. Thus,

$$\int_{-\infty}^{\infty} H_m\, (\tau_1)\, \varphi_{zz}(\tau - \tau_1)d\tau_1 = \varphi_{zx}(\tau), \qquad \text{for } \tau = 0 \qquad (A.17)$$

Equation A.17 is the famous Wiener-Hopf integral equation.

Strictly speaking, the $H_m(\tau)$ that satisfies Eq. A.17 produces a stationary value of the mean square error and does not necessarily ensure a minimum. It can be shown that the solution of Eq. A.17 does yield a minimum, by considering the second derivative of the mean square error with respect to δ. Differentiating both sides of Eq. A.13 yields

$$\frac{d^2\left[\overline{e^2(t)}\right]}{d\delta^2} = 2\int_{-\infty}^{\infty} H_\delta(\tau)d\tau \int_{-\infty}^{\infty} H_\delta(\tau_1)\, \varphi_{zz}(\tau - \tau_1)d\tau_1$$

This is the same as twice the mean square value of the noisy input signal filtered by a weighting function, $H_\delta(\tau)$. Since this mean square signal can never be negative, this shows that the second derivative of the mean square error is always positive, hence the solution of Eq. A.17 corresponds to a minimum.

APPENDIX B

DEVELOPMENT OF THE BODE-SHANNON SOLUTION

Bode and Shannon used frequency domain techniques to solve the filter problem by converting the actual signal power spectral density to that of white noise, and then operating on this white noise signal. The solution may be described by considering Figure 1; $x(t)$ is the signal, $v(t)$ is white noise, $\hat{x}(t)$ is the estimate of the signal, and $e(t)$ is the error. The assumptions are that, (1) input signal and noise are uncorrelated and stationary, and (2) all time functions are Fourier transformable.

The impulse response of the filter, $h(t)$, is chosen on the basis of minimum mean square error. The error signal is defined as

$$e(t) = \hat{x}(t) - x(t) \tag{B.1}$$

and its Fourier transform is,

$$\begin{aligned} E(\omega) &= \hat{X}(\omega) - X(\omega) \\ &= \left[X(\omega) + V(\omega)\right] h(\omega) - X(\omega) \\ &= \left[h(\omega) - 1\right] X(\omega) + h(\omega)V(\omega) \end{aligned} \tag{B.2}$$

Since the signal and noise are uncorrelated, the power spectral density of the error is

$$W_e(\omega) = \frac{1}{2\pi}\left[\left|h(\omega) - 1\right|^2 W_x(\omega) + \left|h(\omega)\right|^2 W_v(\omega)\right] \tag{B.3}$$

Hence the mean square error is

$$\overline{e^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\left|h(\omega) - 1\right|^2 W_x(\omega) + \left|h(\omega)\right|^2 W_v(\omega)\right) d\omega \tag{B.4}$$

This is to be minimized by the proper choice of $h(\omega)$.

Note that to obtain a solution, the power spectral densities $W_x(\omega)$ and $W_v(\omega)$ must be specified. Let

$$h(\omega) = C(\omega)e^{j\theta(\omega)}$$

$$= C(\omega) \quad (\cos\theta(\omega) + j \sin\theta(\omega)) \tag{B.5}$$

Substituting into Eq. B.1 and simplifying yields,

$$\overline{e^2(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ C^2(\omega)W_v(\omega) + \left[C^2(\omega) + 1 - 2C(\omega)\cos(\theta(\omega)) \right] \right\} W_x(\omega)d\omega \tag{B.6}$$

Since $C(\omega)$, $W_x(\omega)$, and $W_v(\omega)$ are nonnegative this expression is minimized when $\cos\theta(\omega)$ has its largest value; that is, when $\theta(\omega)$ equals zero. Thus,

$$\overline{e^2_{opt}(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[C^2(\omega) \left(W_v(\omega) + W_x(\omega) \right) - 2C(\omega)W_x(\omega) + W_x(\omega) \right] d\omega \tag{B.7}$$

In order to find the $C(\omega)$ that minimizes this expression, complete the square of the terms in the bracket. Thus,

$$\overline{e^2_{opt}(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\left(C(\omega)\sqrt{W_v(\omega) + W_x(\omega)} - \frac{W_x(\omega)}{\sqrt{W_v(\omega) + W_x(\omega)}} \right)^2 + \frac{W_x(\omega)W_v(\omega)}{W_x(\omega) + W_v(\omega)} \right] d\omega \tag{B.8}$$

Eq. B.8 is minimized when

$$C(\omega) = \frac{W_x(\omega)}{W_x(\omega) + W_v(\omega)} \tag{B.9}$$

with minimum mean square error given by

$$\overline{e^2_{min}}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{W_x(\omega) W_v(\omega)}{W_x(\omega) + W_v(\omega)} \, d\omega \tag{B.10}$$

Since $\theta(\omega)$ equals zero,

$$h(\omega) = C(\omega) = \frac{W_x(\omega)}{W_x(\omega) + W_v(\omega)} \tag{B.11}$$

The impulse response is therefore given by

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{W_x(\omega)}{W_x(\omega) + W_v(\omega)} \, e^{j\omega t} \, d\omega \tag{B.12}$$

which, in general, does not vanish for $t < 0$. Thus Eq. B.11 is generally not physically realizable.

The minimum phase filter which converts a white noise input to a spectral density equal to that of the signal plus noise, must have a transfer function of magnitude

$$B(\omega) = \sqrt{W_x(\omega) + W_v(\omega)}$$

Since Eq. B.11 gives the best noncausal operation on the signal plus noise, the best noncausal operation on white noise inputs requires that,

$$H_\omega(\omega) = \frac{W_x(\omega) \sqrt{W_x(\omega) + W_v(\omega)}}{W_x(\omega) + W_v(\omega)} \tag{B.13}$$

This however, is still noncausal. The causal filter is then constructed as follows:

$$g_\omega(t) = \begin{cases} 0 & , \ t < 0 \\ h_\omega(t) & , \ t \geq 0 \end{cases} \tag{B.14}$$

The desired causal filter for operation on signal plus noise inputs then has a transfer function given by:

$$H(\omega) = \frac{G_{\omega}(\omega)}{B(\omega)} = G_{\omega}(\omega) \frac{1}{\sqrt{W_x(\omega) + W_v(\omega)}} \tag{B.15}$$

APPENDIX C

POWER SPECTRAL DENSITY PROGRAM

```
      REAL*8 ITITLE(12)
C  FIRST DIMENSION OF INPUT ARRAY TO HARM MUST
C  BE A POWER OF TWO
      COMPLEX*8 A(2048,1,1)
      DIMENSION S(1024),INV(1024)
      DIMENSION M(3),X(4100),B(200),BX(200),TIMEX(200)
      KK=11
      T=0.01
      PIT=6.2831853
      QD=1.0/T
      DEV=SQRT(QD)
      R=1.0/T
      DEX=SQRT(R)
      PHI=EXP(-T)
      GAMMA=1.0-PHI
      IX=409734751
      CALL GAUSS(IX,DEV,0.0,W)
      X(1)=GAMMA*W
      M(1)=KK
      M(2)=0
      M(3)=0
      N1=2**KK
      N2=1
      N3=1
      DO 2 I1=1,N1
      DO 2 I2=1,N2
      DO 2 I3=1,N3
    2 A(I1,I2,I3)=(0.0,0.0)
C    SIGNAL IS NOW GENERATED
      DO 3 I1=1,N1
      DO 3 I2=1,N2
      DO 3 I3=1,N3
      CALL GAUSS(IX,DEV,0.0,W)
      X(I1+1)=PHI*X(I1)+W*GAMMA
    3 A(I1,I2,I3)=X(I1)
C  CALCULATE MEAN AND MEAN SQUARE OF SIGNAL
      AX=0.0
      BZ=0.0
      DO 50 I=1,N1
   50 AX=AX+X(I)
      XBAR=AX/N1
      DO 51 I=1,N1
   51 BZ=BZ+(X(I)-XBAR)**2
      XMS=BZ/N1
      WRITE(6,92)XBAR
   92 FORMAT(T6,'SIGNAL AVERAGE VALUE = ',T32,E13.6)
      WRITE(6,95)XMS
   95 FORMAT(T6,'MEAN SQUARE SIGNAL = ',T30,E13.6)
C  OBTAIN THE FAST FOURIER TRANSFORM
      CALL HARM(A,M,INV,S,1,IFERR)
      NX=35
C  NORMALIZE AND PREPARE INPUTS FOR PLOTTING
      DO 42 I=1,NX
      B(I)=CABS(A(I,1,1))**2
      BX(I)=(B(I)*2.0)/(N1*QD)
      TIMEX(I)=((I-1)*PIT)/(N1*T)
      WRITE(6,63)TIMEX(I),BX(I)
   63 FORMAT(T6,E13.6,T20,E13.6)
   42 CONTINUE
      READ(5,105)(ITITLE(I),I=1,12)
  105 FORMAT(6A8)
      CALL DRAW(NX,TIMEX,BX,0,0,'    ',ITITLE,0.0,0.0,0,0,0,0,8,9,1,LA)
      END
```

APPENDIX D

WIENER-KOLMOGOROV PROGRAM

WIENER-KOLMOGOROV FILTER PROGRAM

```
FORTRAN IV G LEVEL 0, MOD 0                MAIN                DATE = 67330          14/00/57

              REAL*8 ITITLE(12)
              DIMENSION X(3100),XHAT(3100),E(3100),Z(3100),TIME(3100)
              DIMENSION EMS(3100),YXX(310),ES(3100)
              N=3000
              NA=N+5
              IX=409734751
              T=0.01
              QD=1.0/T
              DEV=SQRT(QD)
              R=1.0/T
              DEX=SQRT(R)
              EMS(1)=0.0
              PHI=EXP(-T)
              GAMMA=1.0-PHI
              ROT=SQRT(2.0)
              XPO=ROT*T
              PHK=EXP(-XPO)
              GAMB=1.0-PHK
              WRITE(6,20)
           20 FORMAT(T6,'K',T16,'X(K)',T36,'XHAT(K)',T58,'Z(K)',T80,'EMS(K)')
              XHAT(1)=0.0
              CALL GAUSS(IX,DEV,0.0,W)
              X(1)=GAMMA*W
      C       GENERATE SIGNAL
              DO 10 K=1,NA
              CALL GAUSS(IX,DEV,0.0,W)
              X(K+1)=PHI*X(K)+W*GAMMA
           10 CONTINUE
      C       ADD MEASUREMENT NOISE AND
      C       COMPUTE ESTIMATE OF SIGNAL
              DO 11 K=1,NA
              CALL GAUSS(IX,DEX,0.0,V)
              Z(K)=X(K)+V
              XHAT(K+1)=PHK*XHAT(K)+GAMB*Z(K)*(1.0/(2.0+ROT))
              E(K)=XHAT(K)-X(K)
           11 CONTINUE
              KT=2
      C       COMPUTE MEAN AND MEAN SQUARE ERROR
              DO 85 K=1,N
              YA=0.0
              YB=0.0
              DO 25 I=1,KT
              YA=YA+E(I)
           25 CONTINUE
              YBAR=YA/KT
              DO 26 I=1,KT
              YB=YB+(E(I)-YBAR)**2
           26 CONTINUE
              EMS(K+1)=YB/KT
              KT=KT+1
              WRITE(6,30)K,X(K),XHAT(K),Z(K),EMS(K)
           30 FORMAT(2X,I4,5X,E13.6,9X,E13.6,9X,E13.6,9X,E13.6)
           85 CONTINUE
      C       PREPARE ITEMS FOR PLOTTING
              DO 70 I=1,N
              TIME(I)=I-1
           70 CONTINUE
              DO 110 I=1,N
          110 ES(I)=E(I)**2
              KQ=1
              NPT=300
              DO 43 I=1,NA,10
              YXX(KQ)=EMS(I)
           43 KQ=KQ+1
              READ(5,101)(ITITLE(I),I=1,12)
          101 FORMAT(6A8)
              CALL DRAW(NPT,TIME,YXX,0,0,'     ',ITITLE,0.0,0.0,0,0,0,0,8,9,1,LA)
              END
```

APPENDIX E

KALMAN PROGRAM

```
              REAL*8 ITITLE(12)
              REAL*8 KTITLE(12)
              DIMENSION X(3100),XHAT(3100),E(3100),Z(3100),TIME(3100),EMS(3100)
              DIMENSION YXX(310),GX(310),PK(3100),P(3100),G(3100),ES(3100)
              T=0.01
              N=3000
              NA=N+5
              QD=1.0/T
              DEV=SQRT(QD)
              R=1.0/T
              DEX=SQRT(R)
              IX=409734751
              EMS(1)=0.0
              PHI=EXP(-T)
              GAMMA=1.0-PHI
              XHAT(1)=0.0
              Q=GAMMA*GAMMA*(DEV**2)
              P(1)=10.0
              NB=NA+2
        C     GENERATE GAIN SEQUENCE
              DO 57 K=1,NB
              G(K)=P(K)/(P(K)+R)
              PK(K)=(1.0-G(K))*P(K)
           57 P(K+1)=PHI*PHI*PK(K)+Q
              WRITE(6,20)
           20 FORMAT(T6,'K',T13,'X(K)',T32,'XHAT(K)',T50,'Z(K)',T70,'EMS(K)',T90
             1,'G(K)')
              CALL GAUSS(IX,DEV,0.0,W)
              X(1)=GAMMA*W
        C     GENERATE SIGNAL
              DO 10 K=1,NA
              CALL GAUSS(IX,DEV,0.0,W)
              X(K+1)=PHI*X(K)+W*GAMMA
           10 CONTINUE
              CALL GAUSS(IX,DEX,0.0,V)
              Z(1)=X(1)+V
        C     ADD MEASUREMENT NOISE
        C     COMPUTE ESTIMATE OF SIGNAL
              DO 11 K=1,NA
              CALL GAUSS(IX,DEX,0.0,V)
              Z(K+1)=X(K+1)+V
              XHAT(K+1)=PHI*XHAT(K)+G(K+1)*(Z(K+1)-PHI*XHAT(K))
              E(K)=XHAT(K)-X(K)
           11 CONTINUE
              KT=2
        C     CALCULATE MEAN AND MEAN SQUARE OF ERROR
              DO 85 K=1,N
              YA=0.0
              YB=0.0
              DO 25 I=1,KT
              YA=YA+E(I)
           25 CONTINUE
              YBAR=YA/KT
              DO 26 I=1,KT
              YB=YB+(E(I)-YBAR)**2
           26 CONTINUE
              EMS(K+1)=YB/KT
              KT=KT+1
              WRITE(6,30)K,X(K),XHAT(K),Z(K),EMS(K),G(K)
           30 FORMAT(2X,I4,5X,E13.6,9X,E13.6,9X,E13.6,9X,E13.6,9X,E13.6)
           85 CONTINUE
        C    PREPARE ITEMS FOR PLOTTING
              DO 70 I=1,N
              TIME(I)=I-1
           70 CONTINUE
              DO 110 I=1,N
          110 ES(I)=E(I)**2
              KQ=1
              NPT=300
              DO 43 I=1,NA,10
              YXX(KQ)=EMS(I)
           43 KQ=KQ+1
              READ(5,101)(ITITLE(I),I=1,12)
```

```
  101 FORMAT(6A8)
      CALL DRAW(NPT,TIME,YXX,0,0,'        ',ITITLE,0.0,0.0,0,0,0,0,8,9,1,LA)
      KL=1
      DO 44 I=1,NA,10
      GX(KL)=G(I)
   44 KL=KL+1
      READ(5,101)(KTITLE(I),I=1,12)
      CALL DRAW(NPT,TIME,GX,0,0,'        ',KTITLE,0.0,0.0,0,0,0,0,8,9,1,LA)
      END
```

APPENDIX F

NO FILTER PROGRAM

NO FILTER PROGRAM

```
FORTRAN IV G LEVEL 0, MOD 0                MAIN              DATE = 67330          13/54/55

              REAL*8 ITITLE(12)
              DIMENSION X(3100),XHAT(3100),E(3100),Z(3100),TIME(3100)
              DIMENSION EMS(3100),YXX(310),W(3100),V(3100),ES(3100)
              N=3000
              NA=N+5
              T=0.01
              QC=1.0/T
              DEV=SQRT(QC)
              R=1.0/T
              DEX=SQRT(R)
              IX=409734751
              X(1)=0.0
              EMS(1)=0.0
              PHI=EXP(-T)
              GAMMA=1.0-PHI
              WRITE(6,20)
           20 FORMAT(T6,'K',T14,'X(K)',T34,'Z(K)',T54,'EMS(K)',T74,'W(K)',T94,
             1'V(K)')
              DO 11 K=1,NA
           11 CALL GAUSS(IX,DEV,0.0,W(K))
      C       GENERATE SIGNAL
      C       ADD MEASUREMENT NOISE AND
      C       COMPUTE ESTIMATE OF SIGNAL
              DO 10 K=1,N
              X(K+1)=PHI*X(K)+W(K+1)*GAMMA
              CALL GAUSS(IX,DEX,0.0,V(K))
              Z(K)=V(K)+X(K)
              E(K)=Z(K)-X(K)
           10 CONTINUE
      C       COMPUTE MEAN AND MEAN SQUARE ERROR
              KT=2
              DO 85 K=1,N
              YA=0.0
              YB=0.0
              DO 25 I=1,KT
              YA=YA+E(I)
           25 CONTINUE
              YBAR=YA/KT
              DO 26 I=1,KT
              YB=YB+(E(I)-YBAR)**2
           26 CONTINUE
              EMS(K+1)=YB/KT
              KT=KT+1
              WRITE(6,30)K,X(K),Z(K),EMS(K),W(K),V(K)
           30 FORMAT(2X,I4,4X,E13.6,7X,E13.6,7X,E13.6,7X,E13.6,7X,E13.6)
           85 CONTINUE
      C       PREPARE ITEMS FOR PLOTTING
              DO 70 I=1,N
              TIME(I)=I-1
           70 CONTINUE
              DO 110 I=1,N
          110 ES(I)=E(I)**2
              KQ=1
              NPT=300
              DO 43 I=1,NA,10
              YXX(KQ)=EMS(I)
           43 KQ=KQ+1
              READ(5,101)(ITITLE(I),I=1,12)
          101 FORMAT(6A8)
              CALL DRAW(NPT,TIME,YXX,0,0,'      ',ITITLE,0.0,0.0,0,0,0,0,8,9,1,LA)
              END
```

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This thesis is concerned with a comparative study of discrete time filters using the theories of Wiener-Kolmogorov, Bode-Shannon, and Kalman, applied to the filtering of a non-stationary random signal in the presence of measurement noise. Programs are developed for the simulation of these systems and signals on a digital computer. Their filtering properties are compared for a random input signal with known steady state characteristics, starting at initial time $t = t_0$. The results show that when the gain of the Kalman filter is equal to its steady state value, the Kalman and Wiener-Kolmogorov filters perform identically. For large initial errors the Kalman filter, with large initial gain, gives the best transient response; for small initial errors the Kalman and Wiener-Kolmogorov filters are essentially equivalent in their transient responses.

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